Connectedness.

Definition. The sets H and K are said to be *mutually separated* if and only if neither contains a point or a limit point of the other.

Definition. The set M is said to be *connected* if and only if it is not the union of two non-empty mutually separated sets.

Theorem 6.1. If M is connected then $\overline{M}$ is connected.

Theorem 6.2. Suppose that U and V are disjoint open sets and the connected set M is such that $M \subset U \cup V$. Then either $M \subset U$ or $M \subset V$.

Theorem 6.3. If M is compact and not connected, then M is the union of two disjoint closed (non-empty) sets.

Theorem 6.4. Suppose that G is a collection of connected sets and that there is a point p that lies in every element of G. Then $\cup G$ is connected.

Theorem 6.5. Suppose that $f : X \to Y$ is a continuous function and M is a connected subset of X. Then $f(M)$ is connected.

Theorem 6.6. Suppose that M is connected, U is an open set and M contains a point in U and a point that is not in U. Then $M \cap Bd(U) \neq \emptyset$.

Exercise 6.7.

a. Show that the real line is connected.

b. Show that the completeness axiom is equivalent to the real line being connected.

Definition. If M is a set and $p \in M$, then the component of M containing p is the set to which x belongs if and only if a connected subset of M contains both p and x.

Theorem 6.8. Let $M \subset X$ be a pointset. Then the relation $x \sim p$ defined by $x \sim p$ iff x is in the component of M containing p is an equivalence relation.

Theorem 6.9. If C is the component of the the set M. Then C is a maximal connected subset of M. [This means that 1. C is connected and 2. C is not a proper subset of any connected subset of M.]