Nowhere Dense Sets.

As usual, we assume all spaces are topological Hausdorff spaces.

Definition. Suppose that X is a topological space. The subset M of X is *nowhere dense* in X means that if U is a non-empty open set in X then there is a non-empty open subset V of U that does not intersect M.

Exercise 7.1 Show that every subset of the reals $\mathbb{R}$ that only has finitely many limit points is nowhere dense in $\mathbb{R}$.

Theorem 7.1. If M is a nowhere dense subset of the topological space X then $\overline{M}$ is nowhere dense in X.

Theorem 7.2. If X is normal and the subset M is nowhere dense in X then if U is an open set in X there is an open set V such that $\overline{V} \subset U$ and $\overline{V} \cap M = \emptyset$.

Definition. The space X is *locally compact* means the for each $p \in X$ there is an open set containing p whose closure is compact.

Theorem 7.3. If X is locally compact, then X is not the union of countable many nowhere dense sets. [Hint: use theorem 7.2 together with theorem 5.7.]

Corollary. A closed interval in the reals is uncountable.

Definition. The subset M of the space X is said to be *perfect* if and only if every point of M is a limit point of M.

Theorem 7.4. There exists a closed perfect nowhere dense subset of the reals.

Lemma to theorem 7.4. A closed perfect subset of the reals is nowhere dense if and only if it does not contain an interval.

Definition. A compact perfect nowhere dense subset of the reals is called a Cantor set.

Corollaries to 7.3. Every Cantor set is uncountable. The Reals is not the union of countably many Cantor sets.

Exercise 7.5. Let $f : X \to Y$ be continuous.

a.) If H is a nowhere dense subset of Y, then is $f^{-1}(H)$ nowhere dense in X? What if f is onto? What if f is 1-1 and onto?

b.) If M is a nowhere dense subset of X, then is $f(M)$ nowhere dense in Y? What if f is onto? What if f is 1-1 and onto?

Definition. Suppose that X is a metric space with metric d. The sequence $\{x_i\}_{i=1}^{\infty}$ is said to be a Cauchy sequence if and only if for each positive number ϵ there exists an integer N so that $d(x_k, x_n) < \epsilon$ for all $k, n > N$.

Definition. Suppose that X is a metric space with metric d. Then the metric d is said to be a complete metric if and only if every Cauchy sequence converges with respect to the metric d.

Exercise 7.7. Show that the standard metric for the real numbers is a complete metric.

Exercise 7.8. Show that there exists a metric for the reals which is not a complete metric but which produces the same topology as the standard metric.

Theorem 7.9. If X is a metric space with a complete metric, then X is not the union of countably many nowhere dense sets.
[Note: this is an important theorem in analysis. It is applied to spaces of functions.]

Exercises 7.10. Show the following:

a. If M' is nowhere dense then M is nowhere dense.

b. If U is open then $\mathrm{Bd}(U)$ is nowhere dense.

c. Suppose $M \subset \mathbb{R}$ is closed. Then $\mathbb{R} - M$ is the union of countably many open connected sets.

Example 7.11. [The tangent disc space.] Let $X = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y \geq 0\}$. We describe a basis $\mathcal{B}$ to generate a topology:

If $(a,b) \in X$, $b > 0$ and $\epsilon > 0$ then $B_\epsilon(a,b) = \{(x,y) \mid (x-a)^2 + (y-b)^2 < \epsilon\} \in \mathcal{B}$.

If $(a,b) \in X$, $b = 0$ and $\epsilon > 0$ then $B_\epsilon(a,0) = \{(x,y) \mid (x-a)^2 + (y-\epsilon)^2 < \epsilon\} \cup \{(a,0)\} \in \mathcal{B}$.

The collection $\mathcal{B}$ consists of the interior of circles centered in the upper half-plane together with the interior of circles, in the upper half-plane, tangent to the x-axis together with the point of tangency (hence the name).

a. Show that the collection $\mathcal{B}$ is a basis for a Hausdorff topology $\mathcal{T}$.

b. Show that $(X, \mathcal{T})$ is regular.

c. Show that $\mathbb{Q} = \{(x,0) | x \text{ is rational }\}$ and $\mathbb{P} = \{(x,0) | x \text{ is irrational }\}$ are disjoint closed sets in $(X, \mathcal{T})$.

d. To prove that $(X, \mathcal{T})$ is not normal we assume that U and V are disjoint open sets containing $\mathbb{Q}$ and $\mathbb{P}$ respectively and attempt to arrive at a contradiction. Toward that end, assume U and V are as defined and for a fixed $\delta > 0$ let $M_\delta = \{(a,0) | B_\epsilon(a,0) \subset V, \epsilon > \delta\}$. Show that M_δ is nowhere dense in the x-axis.