

Topology Notes 09

Product Spaces

Definition. Suppose that each of X and Y is a topological space. Then the topological cross product space denoted by $X \times Y$ is the space whose points are the point pairs: $X \times Y = \{(x, y) | x \in X, y \in Y\}$. A basis for the topology of $X \times Y$ is obtained as follows: if U is open in X and V is open in Y then $U \times V$ is a basis element.

Theorem 9.1. If each of X and Y is Hausdorff, then $X \times Y$ is Hausdorff.

Exercise. Give an example of an open set in a product space that is not a basis element as described above.

Exercise. Let Z be the product space $X \times Y$ and let $p \in X$. Show that the set $\{(p, y) | y \in Y\}$ is homeomorphic to Y .

Exercise. Let $Z = X \times X$. Show that the set $\{(x, x) | x \in X\}$ is homeomorphic to X .

Definition. If Z is the product space $X \times Y$ then the function $\pi_1 : Z \rightarrow X$ is defined by $\pi_1(x, y) = x$ and the function $\pi_2 : Z \rightarrow Y$ is defined by $\pi_2(x, y) = y$; these are called the projection maps.

Theorem 9.2. The projection maps π_1 and π_2 have the following properties.

- a. They are continuous.
- b. They map closed sets onto closed sets.
- c. They map open sets onto open sets.

[Note one of these is deliberately false.]

Theorem 9.3. If S is a topological space and $Z = X \times Y$ then $f : S \rightarrow Z$ is continuous if and only if the functions $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

Theorem 9.4. If each of X and Y is a metric space then $X \times Y$ is a metric space.

Theorem 9.5. If each of X and Y is connected then $X \times Y$ is connected.

Theorem 9.6. If one of X and Y is not connected then $X \times Y$ is not connected.

Theorem 9.7. If each of X and Y is compact then $X \times Y$ is compact.

Note this last theorem is a trivial consequence of the much more difficult theorem that says under very weak conditions (the Axiom of Choice) every set can be well ordered.

Generalized Products.

Definition. Suppose that Γ is an index set and for each $\gamma \in \Gamma$, X_γ is a topological space. Then the topological product space denoted by

$$P = \prod_{\gamma \in \Gamma} X_\gamma \times Y$$

is the space whose points are functions $x : \Gamma \rightarrow \cup_{\gamma \in \Gamma} X_\gamma$ so that $x(\gamma) \in X_\gamma$. A basis element B for the topology of P is obtained as follows: if $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ is a finite sequence of distinct index elements then for each i there is an open set U_{γ_i} in X_{γ_i} so that $B = \{x \in P | x(\gamma_i) \in U_{\gamma_i}\}$.

Show the usual relationships between X_γ for all γ and P .

Example A. For each $i \in \mathbb{N}$ let $X_i = \{0, 1\}$ with the discrete topology. Then the space

$$P_1 = \prod_{i=1}^{\infty} X_i$$

is a compact perfect metric space with a basis of closed sets.

Example B. For each $i \in \mathbb{N}$ let $X_i = [0, 1]$ with the usual topology. Then the space

$$P_2 = \prod_{i=1}^{\infty} X_i$$

is a compact connected metric space.

Topological Groups.

Definition. Suppose that X is a topological space and $*$ is an operation on X . Then $*$ is said to be continuous if the function $\varphi : X \times X \rightarrow X$ defined by $\varphi(x, y) = x * y$ is continuous.

Definition. Suppose that X is a topological space and $*$ is an operation on X so that $(X, *)$ is a group. Then X is called a topological group if the group operation is continuous.

Exercise

a. If C is a subset of the complex plane $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ is defined by $C = \{a+bi \mid a^2+b^2 = 1\}$ and $*$ is complex multiplication. Then, with the operation $*$, C is a topological group.

b. Suppose that for $X_i = \{0, 1\}$ we have the $*_i$ denote that mod 2 addition. Then with the termwise mod 2 operation defined on example A, the space P_1 (as defined above) is a topological group.