MATH5630/6630 Dr. Smith Test 1, June 17, 2022.

Make sure to show all your work. You may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled and will assume that the scratch work is not part of the final answer. You may use your calculators.

Problem 1.

a.) Set up the iteration formula that uses Newton's Method for calculating roots to calculate the fifth root of a number.

Solution.

$$x_{n+1} = x_n - \frac{x_n^5 - 3}{5x_n^4}$$

□

b.) Use the iteration formula from above (Newton's Method) to estimate $\sqrt[5]{3}$; do two iterations starting with $x_0 = 2$.

Solution.

$$\begin{aligned} x_1 &= \frac{131}{80} = 1.6375 \\ x_2 &= 1.39345 \end{aligned}$$

□

Problem 2. A mathematician in ancient Babylon wants to estimate the solution of the equation (in modern formulation) $x^2 + x = 1$. He solves for x as follows:

$$\begin{aligned} x^2 + x &= 1 \\ x(x+1) &= 1 \\ x &= \frac{1}{x+1}. \end{aligned}$$

Then he guesses a solution and iterates using this last formula. Will his method work (in other words, if he makes a reasonable first guess, will the iterations limit to the root of the equation.) [Note that the Babylonians did not know about negative numbers, so he's looking for the positive root.] You may use your (modern) knowledge about the solution to answer the problem.

Solution. This is an iteration fixed point problem where $g(x) = \frac{1}{x+1}$ and so we want x^* so that $g(x^*) = x^*$. The root of $x^2 + x - 1 = 0$ is

$$\begin{aligned} x^* &= \frac{-1 \pm \sqrt{5}}{2} \\ x^* &\approx 0.61802. \end{aligned}$$

And we have

$$\begin{aligned} g'(x) &= \frac{-1}{(x+1)^2} \\ g'(x*) &= \frac{-1}{(1.61802\ldots)^2} \\ |g'(x)| &< 1. \end{aligned}$$

Therefore the iteration process converges. [Note: actually a quick examination of g' gets that $|g'(x)| < 1$ for all values of $x > 0$ and that's sufficient for the problem without calculating the root.] □

Problem 3. Give the Lagrange form of the best polynomial that interpolates the following data points:

x	0.5	0.75	0.9	1.0
y	1.0	1.2	1.5	2.0

Solution.

$$\begin{aligned} p(x) &= 1.0\frac{(x-0.75)(x-0.9)(x-1)}{(0.5-0.75)(0.5-0,9)(0.5-1)} + 1.2\frac{(x-0.5)(x-0.9)(x-1)}{(0.75-0.5)(0.75-0.9)(0.75-1)} \\ &\quad +1.5\frac{(x-0.5)(x-0.75)(x-1)}{(0.9-0.5)(0.9-0.75)(0.9-1)} + 2.0\frac{(x-0.5)(x-0.75)(x-.9)}{(1.0-0.5)(1.0-0.75)(1.0-0.9)} \\ &= -20(x-0.75)(x-0.9)(x-1) + 128(x-0.5)(x-0.9)(x-1) + \\ &\quad -250(x-0.5)(x-0.75)(x-1) + 160(x-0.5)(x-0.75)(x-.9). \end{aligned}$$

□

Problem 4. Given the following data at the two points, just give the form of the best interpolation polynomial and set up the equations for the coefficients:

x	1.2	1.5
$f(x)$	2.3	3.1
$f'(x)$	0.5	0.4
$f''(x)$	-2	2.5

Solution. There are six data points so six equations are needed to find six constants, so the polynomial will be of degree five.

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5.$$

The equations are:

$$\begin{aligned}
c_0 + c_1(1.2) + c_2(1.2)^2 + c_3(1.2)^3 + c_4(1.2)^4 + c_5(1.2)^5 &= 2.3 \\
c_0 + c_1(1.5) + c_2(1.5)^2 + c_3(1.5)^3 + c_4(1.5)^4 + c_5(1.5)^5 &= 3.1 \\
c_1 + 2c_2(1.2) + 3c_3(1.2)^2 + 4c_4(1.2)^3 + 5c_5(1.2)^4 &= 0.5 \\
c_1 + 2c_2(1.5) + 3c_3(1.5)^2 + 4c_4(1.5)^3 + 5c_5(1.5)^4 &= 0.4 \\
2c_2 + 6c_3(1.2) + 12c_4(1.2)^2 + 20c_5(1.2)^3 &= -2 \\
2c_2 + 6c_3(1.5) + 12c_4(1.5)^2 + 20c_5(1.5)^3 &= 2.5.
\end{aligned}$$

Some people used the Newton's form and that's fine; the equations are just different.

□

Problem 5. Write out the form of the cubic splines and set up the equations for the coefficients to construct the cubic spline through the following three points:

x	2.1	2.5	3.0
y	4.0	4.3	5.1

For the extra conditions to complete the number of equations, assume that the derivative at the first point is equal to the slope of the line connecting the first two points and that the derivative at the third point is equal to the slope of the line connecting the last two points.

Solution. The two cubic splines will look like:

$$\begin{aligned} s_0(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 \\ s_1(x) &= d_0 + d_1 x + d_2 x^2 + d_3 x^3. \end{aligned}$$

The equations are:

$$\begin{aligned} c_0 + c_1(2.1) + c_2(2.1)^2 + c_3(2.1)^3 &= 4.0 \\ c_0 + c_1(2.5) + c_2(2.5)^2 + c_3(2.5)^3 &= 4.3 \\ d_0 + d_1(2.5) + d_2(2.5)^2 + d_3(2.5)^3 &= 4.3 \\ d_0 + d_1(3.0) + d_2(3.0)^2 + d_3(3.0)^3 &= 5.1 \\ c_1 + 2c_2(2.5) + 3c_3(2.5)^2 &= d_1 + 2d_2(2.5) + 3d_3(2.5)^2 \\ 2c_2 + 6c_3(2.5) &= 2d_2 + 6d_3(2.5) \\ c_1 + 2c_2(2.1) + 3c_3(2.1)^2 &= \frac{3}{4} = 0.75 \\ d_1 + 2d_2(3) + 3d_3(3)^2 &= \frac{8}{5} = 1.6. \end{aligned}$$

□

Problem 6. Consider the following Taylor series approximation for the sin function:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

In order to calculate sin(0.2), what is the best estimate that you can determine for the error using these terms of the sine function Taylor series; base your calculation on the assumption that the sine and cosine functions are bounded between -1 and 1?

Solution. The form of the remainder is

$$R = \sin^{[7]}(\xi)\frac{x^7}{7!}$$

where $0 < \xi < 0.2$ and $0 < x < 0.2$. Recall that the even terms of the Taylor series for the sine function are 0 so there is a $0 \cdot \frac{x^6}{6!} = 0$ term. Some students used the remainder term with the sixth as opposed to the seventh derivative of the sine function (for a small loss of points.) Using the fact that the since the sine and cosine are bounded between -1 and 1, the numbers for the magnitude of the best error estimate error are:

$$\begin{aligned}\frac{(0.2)^7}{7!} &\approx 2.5397 \times 10^{-9} \\ \frac{(0.2)^6}{6!} &\approx 8.889 \times 10^{-8}.\end{aligned}$$

$\square$