MATH5630/6630 Dr. Smith, Test 2, July 22, 2022.

Make sure to show all your work. You may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled and will assume that the scratch work is not part of the final answer. You may use your calculators. Note the sheet of "Helpful Formulas."

Problem 1. Use Euler's method to solve the following system of differential equations. Use the increment $h = 0.01$; use the initial condition $x(0) = 2, y(0) = 1.5$:

$$\begin{aligned} x'(t) &= 3x(t) - 5y(t) \\ y'(t) &= y(t) - (x(t))^2. \end{aligned}$$

Do two iterations.

Solution.

$$\begin{aligned} x_{i+1} &= x_i + h(3x_i - 5y_i) \\ y_{i+1} &= y_i + h(y_i - x_i^2). \end{aligned}$$

i	h	x_i	y_i
0		2	1.5
1	0.01	1.985	1.475
2	0.01	1.9708	1.45034775

□

Problem 2. Use the backwards Euler's method to solve the following differential equation. Use the increment $h = 0.01$; use the initial condition $y(0) = 2$:

$$y'(x) = 3x - 5y(x).$$

Do two iterations.

Solution.

$$\begin{aligned} y_{i+1} &= y_i + h(f_{i+1}) \\ y_{i+1} &= y_i + h(3x_{i+1} - 5y_{i+1}) \\ y_{i+1}(1+5h) &= y_i + 3hx_{i+1} \\ y_{i+1} &= \frac{y_i + 3hx_{i+1}}{1+5h}. \end{aligned}$$

i	h	x_i	y_i
0	0.01	0	2
1	0.01	0.01	1.905047619
2	0.01	0.02	1.814902494

□

Problem 3. Use the two step method to solve the following differential equation. Use the increment $h = 0.01$; use the initial condition $y(0) = 2$:

$$y'(x) = 3x - 5y(x).$$

Do two iterations, and justify your choice for y_{-1}.

Solution.

$$\begin{aligned} y_{i+1} &= y_i + \frac{h}{2}(3f_i - f_{i-1}) \\ y_{i+1} &= y_i + \frac{h}{2}(3(3x_i - 5y_i) - (3x_{i-1} - 5y_{i-1})) \end{aligned}$$

i	h	x_i	y_i
-1	0.01	-0.01	2.1
0	0.01	0	2
1	0.01	0.01	1.90265
2	0.01	0.02	1.81040125

□

Problem 4. Consider the following function defined on $[-\pi, \pi]$.

$$f(x) = \pi^2 - x^2.$$

i.) Just set up the integrals that evaluate the Fourier coefficients $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$.

Solution.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)\cos(nx)dx, \quad n = 0, 1, 2, \ldots$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)\sin(nx)dx, \quad n = 1, 2, \ldots$$

Some students may have $b_n = 0$, that's fine; some students may have $\frac{a_0}{2}$ separately:

$$\frac{a_0}{2} = \frac{1}{2} \times \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)dx.$$

□

ii.) Determine which coefficients are 0.

Solution. The b_n's are 0. □

iii.) Calculate the constant term $\frac{a_0}{2}$.

Solution.

$$\frac{a_0}{2} = \frac{2}{3}\pi^2.$$

□

Problem 5. Calculate the Fourier Series (find all the a_n's and b_n's) for the following function f defined over the interval $[-\pi, \pi]$:

$$f(x) = 3\cos(5x) - 7.5\sin(4x).$$

Solution. Since

$$\int_{-\pi}^{\pi} \cos(5x)\cos(kx)dx = 0$$

whenever $k \neq 5$, we have $a_n \neq 0$ only if $n = 5$. So

$$\begin{aligned} \frac{1}{\pi}\int_{-\pi}^{\pi} 3\cos(5x)\cos(5x)dx &= 3\frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(5x)dx \\ &= 3 \cdot 1. \end{aligned}$$

Similarly for the $\sin(4x)$ portion. So the fourier series for the function is the function itself. Thus we have $a_5 = 5$ and $b_4 = -7.5$ and the rest of the coefficients are zero. □

Problem 6i. Show that the following set of functions defined over the $[-L, L]$ is an orthogonal collection defined over this interval:

$$\left\{ \cos\left(\frac{n\pi x}{L}\right) \right\}_{n=1}^{\infty}.$$

Solution. If $k \neq n$ then

$$\begin{aligned} \int_{-L}^{L} \cos\left(\frac{k\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx &= \int_{-L}^{L} \cos\left(\frac{(k-n)\pi x}{L}\right) + \cos\left(\frac{(k+n)\pi x}{L}\right)dx \\ &= \frac{L}{(k-n)\pi}\sin\left(\frac{(k-n)\pi x}{L}\right) + \\ &\quad + \frac{L}{(k+n)\pi}\sin\left(\frac{(k+n)\pi x}{L}\right)\Big|_{-L}^{L} \end{aligned}$$

and since sin is 0 for any integer multiple of π, this integral is zero. □

6ii. Evaluate the following integrals needed to "normalize" these functions:

$$\int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right)dx.$$

Solution. If $k \neq n$ then

$$\begin{aligned} \int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right)dx &= \int_{-L}^{L} \frac{1}{2}\Big(1 + \cos\left(\frac{2n\pi x}{L}\right)\Big) \\ &= \frac{1}{2}\Big(x + \left(\frac{L}{2n\pi}\right)\sin\left(\frac{2n\pi x}{L}\right)\Big)\Big|_{-L}^{L} \\ &= \frac{1}{2}(L - -L + 0 - 0) = L. \end{aligned}$$

□

Extra Credit: Derive the iteration formula for the two step method used to solve a differential equation numerically (see formula sheet).

Helpful Formulas.

1.) Backwards Euler:

$$y_{i+1} = y_i + hf_{i+1}.$$

2.) Two step multistep method:

$$y_{i+1} = y_i + \frac{h}{2}(3f_i - f_{i-1}).$$

3.) Some trig formulas:

$$\begin{aligned}
\sin(\alpha)\sin(\beta) &= \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \\
\cos(\alpha)\cos(\beta) &= \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)] \\
\sin(\alpha)\cos(\beta) &= \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)].
\end{aligned}$$