MATH5630/6630 Dr. Smith Test 1, June 16, 2023.

Make sure to show all your work. You may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled and will assume that the scratch work is not part of the final answer. You may use your calculators.

Problem 1.

a. Use the Newton iteration method to estimate the solution to the equation $3x^3 + 2x - 1 = 0$. Start with $x_0 = 2$ and do two iterations.

Solution. Newton's iteration formula:

$$x_{n=1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Note the minus sign. Two iterations yield:

$$\begin{aligned} x_0 &= 2 \\ x_1 &= 1.2895. \\ x_2 &= 0.8172 \end{aligned}$$

□

b. Argue that the equation only has one root.

Solution. Note that

$$f'(x) = 9x^2 + 2$$

which is always positive; so, since the derivative is positive, the function is increasing and so can intersect the x-axis only once.

□

Problem 2. Give the Lagrange form of the best polynomial that interpolates the following data points:

x	0.5	0.75	1.5	2
y	1.0	2	1.5	.8

Solution.

$$\begin{aligned}
p(x) &= 1\cdot\frac{(x-.75)(x-1.5)(x-2)}{(0.5-0.75)(0.5-1.5)(0.5-2)} + 2\cdot\frac{(x-0.5)(x-1.5)(x-2)}{(0.75-0.5)(0.75-1.5)(0.75-2)}\\
&\quad +1.5\cdot\frac{(x-0.5)(x-.75)(x-2)}{(1.5-0.5)(1.5-1.5)(1.5-2)} + .8\cdot\frac{(x-0.5)(x-1.5)(x-0.75)}{(2-0.5)(2-0.75)(2-1.5)}.
\end{aligned}$$

Some people multiplied it out, that wasn't necessary, but here're my calculations:

$$\begin{aligned}
p(x) &= 1\cdot\frac{(x-.75)(x-1.5)(x-2)}{-0.375} + 2\cdot\frac{(x-0.5)(x-1.5)(x-2)}{0.23444}\\
&\quad +1.5\cdot\frac{(x-0.5)(x-.75)(x-2)}{-3.75} + .8\cdot\frac{(x-0.5)(x-1.5)(x-0.75)}{0.9375}\\
&= -2.6667\cdot(x-.75)(x-1.5)(x-2) + 8.5333\cdot(x-0.5)(x-1.5)(x-2)\\
&\quad + -4\cdot(x-0.5)(x-.75)(x-2) + 0.8533\cdot(x-0.5)(x-1.5)(x-0.75).
\end{aligned}$$

□

Problem 3. Set up the divided difference table for the data points of problem 2 and use it to find the Newton form of the interpolating polynomial that contains them.

Solution.

$$\begin{array}{lllll}
0.7 & 1 & & & \\
0.75 & 2 & 4 & & \\
1.5 & 1.5 & -0.6667 = -\frac{2}{3} & -4.6667 = -\frac{14}{3} & \\
2 & 0.8 & -1.4 & -0.58667 = -\frac{44}{75} & 2.72
\end{array}$$

$$\begin{aligned}
p(x) &= 1 + 4(x-0.5) - 4.6667(x-0.5)(x-0.75)\\
&\quad +2.72(x-0.5)(x-0.75)(x-1.5)
\end{aligned}$$

□

Problem 4. Write out the form of the cubic splines and set up the equations for the coefficients to construct the cubic spline through the following three points; use the "natural" method that assumes that f'' is 0 at the endpoints.

x	2.1	2.5	3.0
y	4.0	4.3	5.1

Solution.

$$\begin{aligned} s_0 &= a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3, \quad x \in [2.1, 2.5] \\ s_1 &= a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3, \quad x \in [2.5, 3.0]. \end{aligned}$$

Substituting and differentiating as needed and substituting again yields:

$$\begin{aligned}
s_0(x_0) = y_0 \;&\Rightarrow\; 4 = a_0 && (1) \\
s_0(x_1) = y_1 \;&\Rightarrow\; 4.3 = a_0 + b_0(0.4) + c_0(0.4)^2 + d_0(0.4)^3 && (2) \\
s_1(x_1) = y_1 \;&\Rightarrow\; 4.3 = a_1 && (3) \\
s_1(x_2) = y_2 \;&\Rightarrow\; 5.1 = a_1 + b_1(0.5) + c_1(0.5)^2 + d_1(0.5)^3 && (4) \\
&\; s_0'(x) = b_0 + 2c_0(x - x_0) + 3d_0(x - x_0)^2 \\
&\; s_1'(x) = b_1 + 2c_1(x - x_1) + 3d_1(x - x_1)^2 \\
s_0'(x_1) = s_1'(x_1) \;&\Rightarrow\; b_0 + 2c_0(0.4) + 3d_0(0.4)^2 = b_1 && (5) \\
&\; s_0''(x) = 2c_0 + 6d_0(x - x_0) \\
&\; s_1''(x) = 2c_1 + 6d_1(x - x_1) \\
s_0''(x_1) = s_1''(x_1) \;&\Rightarrow\; 2c_0 + 6d_0(0.4) = 2c_1 && (6) \\
s_0''(x_0) = 0 \;&\Rightarrow\; 2c_0 = 0 && (7) \\
s_1''(x_2) = 0 \;&\Rightarrow\; 2c_1 + 6d_1(0.5) = 0. && (8)
\end{aligned}$$

□

Problem 5. Consider the following Taylor series approximation for the $f(x) = e^x$ function:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}.$$

a. What is the form of the remainder from Taylor's theorem.

Solution.

$$\text{error} = \frac{f^{[n+1]}(\xi)}{(n+1)!} x^{n+1}, \quad \xi \in [0, x].$$

□

b. Use part (a.) to determine the error in using these terms to approximate $e^{0.2}$; estimate that error based on the assumption that $e < 4$.

Solution.

$$\begin{aligned}
\text{error} &= \frac{e^{\xi}}{5!}(0.2)^5, \quad \xi \in [0, 0.2] \\
&< \frac{4^{0.2}}{5!}(0.2)^5 \\
&< \frac{4}{5!}(0.2)^5 = 1.0667 \times 10^{-5}.
\end{aligned}$$

Some students calculated $4^{0.2} \approx 1.3195$ to obtain 3.5187×10^{-6}. I used 4 because the quantity $4^{0.2}$ needs to be calculated using $e^{0.2}$ or some series equivalent to the one given; in any case I gave full credit for this. □

Problem 6. Consider the following function.

$$f(x) = \begin{cases} \pi & \text{if } 0 \le x \le \pi \\ 2\pi - x & \text{if } \pi < x \le 2\pi \end{cases}$$

a. Set up the integrals that calculate the coefficients of the Fourier series for the function over the interval $[0, 2\pi]$. (Make sure the notation $f(x)$ is not part of your answer.)

Solution.

$$\begin{aligned}
a_k &= \frac{1}{\pi}\Big[\int_0^{\pi} \pi\cos(kx)dx + \int_{\pi}^{2\pi}(2\pi - x)\cos(kx)dx\Big], \quad k = 0, 1, 2, \ldots \\
b_k &= \frac{1}{\pi}\Big[\int_0^{\pi} \pi\sin(kx)dx + \int_{\pi}^{2\pi}(2\pi - x)\sin(kx)dx\Big], \quad k = 0, 1, 2, \ldots.
\end{aligned}$$

□

b. Calculate the constant term (denoted by $\frac{a_0}{2}$ in class and in the textbook).

Solution. Let c denote the constant term ($c = \frac{a_0}{2}$).

$$\begin{aligned} c\int_0^{2\pi} 1dx &= \int_0^{2\pi} f(x)dx \\ c \cdot 2\pi &= \pi^2 + \frac{1}{2}\pi^2 \\ c \cdot 2\pi &= \frac{3}{2}\pi^2 \\ c &= \frac{3}{4}\pi. \end{aligned}$$

The integral on the right is equal to the area under the curve from $x = 0$ to $x = 2\pi$ which is a square with side π (area $= \pi^2$) plus the triangle that's half the square (area $= \frac{1}{2}\pi^2$). $\square$

Extra Credit: Evaluate a_1 and b_1.