MATH5630/6630 Dr. Smith Test 2, July 21, 2023.

Please show all your work; you may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled and will assume that the scratch work is not part of the final answer. You may use your calculators.

Problem 1. We are interested in the estimation of the second derivative $f''(a)$ using at most the three function values: $f(a), f(a+h), f(a-h)$. [Note: it's not necessary to use all three.]

(a.) Derive the formula.

(b.) Use the formula to estimate $f''(8)$ for $f(x) = \sqrt[3]{x}$; use $h = 0.1$.

(c.) What is the order of the error term, in the form $O(h^k)$. [I.e.: what's the value of k that indicates the order of the error term.]

Solution. (a.)

$$\begin{aligned}
f(a+h) &= f(a) + f'(a)h + f''(a)\frac{h^2}{2} + O(h^3) \\
f(a-h) &= f(a) - f'(a)h + f''(a)\frac{h^2}{2} + O(h^3) \\
f(a+h) + f(a-h) &= 2f(a) + f''(a)h^2 + O(h^3) \\
f''(a) &= \frac{f(a+h) + f(a-h) - 2f(a)}{h^2} + O(h).
\end{aligned}$$

Or

$$\begin{aligned}
f(a+h) &= f(a) + f'(a)h + f''(a)\frac{h^2}{2} + f'''(a)\frac{h^3}{6} + O(h^4) \\
f(a-h) &= f(a) - f'(a)h + f''(a)\frac{h^2}{2} - f'''(a)\frac{h^3}{6} + O(h^4) \\
f(a+h) + f(a-h) &= 2f(a) + f''(a)h^2 + O(h^4) \\
f''(a) &= \frac{f(a+h) + f(a-h) - 2f(a)}{h^2} + O(h^2).
\end{aligned}$$

(b.) $f''(8) \approx -0.00694485$.

(c.) $O(h)$ if you used the first derivation,$O(h^2)$ if you used the second. □

Problem 2. Use the trapezoid method to approximate the following integral:

$$\int_2^3 \sqrt{x+1}dx$$

.

(a.) Do the calculation with $h = 0.5$.
(b.) Do the calculation with $h = 0.25$.
(c.) Use the technique developed in class to estimate the error in the approximation obtained in (b.) above.

Solution.

$$\begin{array}{lrcl} \text{(a.)} & T_{0.5} & = & 1.86843 \\ \text{(b.)} & T_{0.25} & = & 1.86903 \\ \text{(c.)} & \text{error} & = & \frac{1}{3}|T_{0.5} - T_{0.25}| = 0.0002011 \end{array}$$

$\square$

Problem 3. A mathematician is using a quadrature technique to calculate an integral $\int_1^2 f(x)dx$. When using a step size of h the error is $Mf'''(\xi)h^3$ (where M is a fixed constant independent of the function or of h):

$$\int_1^2 f(x)dx \quad = \quad Q_h + Mf'''(\xi)h^3.$$

In proceeding he obtains these two calculation (where Q_h is the estimate the method gives for the h value given):

$$\begin{aligned} Q_{0.01} &= 2.2426135 \\ Q_{0.005} &= 2.2429713. \end{aligned}$$

(a.) Derive the formula for estimating the error for the estimate $Q_{\frac{h}{2}}$.
(b.) Use the formula obtained in (a.) to estimate the error for the estimate $Q_{0.005}$.

Solution. (a.) Let I denote the integral $\int_1^2 f(x)dx$. Then

$$\begin{aligned}
I &= Q_h + Mf'''(\xi_1)h^3 \\
I &= Q_{h/2} + Mf'''(\xi_2)\frac{h^3}{8} \\
I &= Q_h + M(f'''(\xi_2) + f''''(\hat{\xi})h)h^3 \\
&\approx Q_h + Mf'''(\xi_2)h^3 \\
0 &\approx Q_h - Q_{h/2} + Mf'''(\xi_2)(1-\frac{1}{8})h^3 \\
\frac{7}{8}Mf'''(\xi_2)h^3 &\approx Q_{h/2} - Q_h \\
\frac{7}{8}Mf'''(\xi_2)h^3 &\approx \frac{1}{7}(Q_{h/2} - Q_h) \\
\text{error} &\approx \frac{1}{7}(Q_{h/2} - Q_h).
\end{aligned}$$

(b.) error $\approx \frac{1}{7}(Q_{h/2} - Q_h) = 0.000051114 = 5.1114e - 5.$ □

Problem 4. Use the Euler method to estimate the solution of the following differential equation:

$$y'(x) = 3x + 1 - (y(x))^2; \quad y(0) = 1.$$

Use $h = 0.1$ and do two iterations.

Solution. The recursion formula is

$$y_{i+1} = y_i + h(3x_i + 1 - y_i^2).$$

i	h	x_i	y_i
0	0.1	0	1
1	0.1	0.1	1
2	0.1	0.2	1.03

□

Problem 5. Use the Euler method to estimate the solution of the following system of two differential equation:

$$
\begin{aligned}
u'(x) &= 3v(x) - 5u(x); \\
v'(x) &= (u(x))^2 + 2v(x); \\
\begin{bmatrix} u(0) \\ v(0) \end{bmatrix} &= \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}
\end{aligned}
$$

Use $h = 0.1$ and do two iterations.

Solution. The recursion formulas are

$$
\begin{aligned}
u_{i+1} &= u_i + h(3v_i - 5u_i) \\
v_{i+1} &= v_i + h(u_i^2 + 2v_i).
\end{aligned}
$$

i	h	x_i	u_i	v_i
0	0.1	0	1.5	1
1	0.1	0.1	1.05	1.425
2	0.1	0.2	0.9525	1.82025

□

Problem 6. Convert the following second order differential equation to a system of two first order differential equations:

$$y''(x) + 5y'(x) - 3xy(x) = 0; \quad y(0) = 1, y'(0) = 2.$$

Solution. Let $u(x) = y'(x)$ and $v(x) = y(x)$ then the second order differential equation becomes the following system:

$$
\begin{aligned}
u'(x) &= -5u(x) + 3xv(x); \\
v'(x) &= u(x); \\
\begin{bmatrix} u(0) \\ v(0) \end{bmatrix} &= \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\end{aligned}
$$

□

Extra Credit. Convert the following third order differential equation to a system of three first order differential equations:

$$y'''(x) - 6y''(x) + (x^2 + 3)y'(x) - 7y(x) = 0; \quad y(0) = 1, y'(0) = 2, y''(0) = 3.$$

Solution. Let $w(x) = y''(x), u(x) = y'(x)$ and $v(x) = y(x)$ then the second order differential equation becomes the following system:

$$\begin{aligned}
w'(x) &= 6w(x) - (x^2+3)u(x) + 7v(x) \\
u'(x) &= w(x); \\
v'(x) &= u(x); \\
\begin{bmatrix} w(0) \\ u(0) \\ v(0) \end{bmatrix} &= \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}
\end{aligned}$$

□