MATH5630/6630 Dr. Smith Test 1, June 14, 2024.

Make sure to show all your work. You may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled and will assume that the scratch work is not part of the final answer. You may use a non-programable calculator.

Problem 1. Consider the function $g(x) = \sqrt[3]{3x-1}$.

a.) Argue that the function has a fixed point between $x = 1$ and $x = 3$.

b.) show that the fixed point satisfies the required conditions to employ the fixed point iteration process to approximate the fixed point.

c.) Use two iterations of the fixed point iteration process to approximate the fixed point; start at $x_0 = 2$.

Solution. Part a.

$$\begin{aligned} g(x) - x &= \sqrt[3]{3x-1} - x \\ g(1) - 1 &= \sqrt[3]{2} - 1 > 0 \\ g(3) - x &= \sqrt[3]{8} - 3 = 2 - 3 < 0. \end{aligned}$$

So there is a root r so that $g(r) - r = 0$ and so $g(r) = r$.

Part b.

$$\begin{aligned} g(x) &= \sqrt[3]{3x-1} \\ g'(x) &= \frac{1}{3}(3x-1)^{-\frac{2}{3}}3 = (3x-1)^{-\frac{2}{3}}. \end{aligned}$$

Observe that if $x > 1$ then $0 < g'(x) < 1$; so the derivative at the fixed point must be less than 1.

Part c.

$$\begin{aligned} g(2) &= \sqrt[3]{5} = 1.7100 \\ g(1.7100) &= 1.6044. \end{aligned}$$

□

Problem 2. Use two iterations of Newton's method to approximate the root of the polynomial $x^5 + 5x - 10$; use $x_0 = 2$ as your first guess.

Solution.

$$\begin{aligned} f(x) &= x^5 + 5x - 10 \\ f'(x) &= 5x^4 + 5 \\ g(x) &= x - \frac{f(x)}{f'(x)} \\ g(2) &= 1.6236 \\ g(1.6236) &= 1.3870. \end{aligned}$$

□

Problem 3. Consider the following data:

i	x_i	y_i
0	1.2	2
1	1.4	2.1
2	1.7	2.05

use a divided difference table to calculate Newton's form of the polynomial that interpolates this data.

Solution. See problem 4 below for the full divided difference table. From the first three lines of the table we have (see problem 4):

$$P(x) = 2 + 0.5(x - 1.2) - 1.3333(x - 1.2)(x - 1.4)$$

□

Problem 4. A fourth piece of data $(x_3, y_3) = (1.8, 2.02)$ was obtained to add to the data of problem 3. Use this new data to modify the Newton's form of the interpolating polynomial.

Solution.

1.2	2.0			
1.4	2.1	0.5		
1.7	2.05	−0.16667	−1.3333	
1.8	2.02	−0.3	−0.3333	1.6667

$$\begin{aligned} P(x) &= 2 + 0.5(x-1.2) - 1.3333(x-1.2)(x-1.4) + \\ &\quad +1.6666(x-1.2)(x-1.4)(x-1.7) \end{aligned}$$

□

Problem 5. A natural cubic spline on $[2,4]$ is defined by

$$s(x) = \begin{cases} s_0(x) = 5 + 3(x-2) - 2(x-2)^2 - (x-2)^3 & 2 \le x < 3 \\ s_1(x) = a + b(x-3) + c(x-3)^2 + d(x-3)^3 & 3 \le x \le 4. \end{cases}$$

Find the values of the unknown quantities; also calculate the original points for which s interpolates (i.e: $s(x)$ at the values $x = 2, 3, 4$).

Solution. Note that I erroneously picked a $s_0(x)$ so that $s_0''(2) \neq 0$. So the spline is not natural at the left end. But all you need to solve the problem is the assumption of natural at the right end.

$$
\begin{aligned}
s_0(x) &= 5 + 3(x-2) - 2(x-2)^2 - (x-2)^3 \\
s_1(3) &= s_0(3) \\
a &= 5 + 3 - 2 - 1 = 5 \\
s_0'(x) &= 3 - 4(x-2) - 3(x-2)^2 \\
s_1'(x) &= b + 2c(x-3) + 3d(x-3)^2 \\
s_1'(3) &= s_0'(3) \\
b &= 3 - 4 - 3 = -4 \\
s_0''(x) &= -4 - 6(x-2) \\
s_1''(x) &= 2c + 6d(x-3) \\
s_1''(3) &= s_0''(3) \\
2c &= -4 - 6 = -10 \\
c &= -5 \\
s_1''(4) &= 0 \\
2c + 6d &= 0 \\
6d &= -2c = 10 \\
d &= \frac{5}{3}.
\end{aligned}
$$

$$
\begin{aligned}
s(2) &= s_0(2) = 5 \\
s(3) &= s_1(3) = a = 5 \\
s(4) &= s_1(4) = a + b + c + d \\
&= 5 - 4 - 5 + \frac{5}{3} = -2\frac{1}{3}.
\end{aligned}
$$

□

Problem 6. Use Taylor's theorem to derive an estimator for the derivative of a function so that the error is at least $O(h^2)$. Include the error term. Then use the derived estimator to estimate the derivative of $\ln(x)$ at $x = 0.5$ use $h = 0.1$.

Solution. Derivation:

$$\begin{aligned}
f(a+h) &= f(a) + f'(a)h + f''(a)\frac{h^2}{2} + f'''(\xi_1)\frac{h^3}{3!} \\
f(a-h) &= f(a) - f'(a)h + f''(a)\frac{h^2}{2} - f'''(\xi_2)\frac{h^3}{3!} \\
f(a+h) - f(a-h) &= 2f'(a)h + (f'''(\xi_1) + f'''(\xi_2))\frac{h^3}{3!} \\
\frac{1}{2h}f(a+h) - f(a-h) &= f'(a) + \frac{1}{2}(f'''(\xi_1) + f'''(\xi_2))\frac{h^2}{3!} \\
\frac{1}{2h}f(a+h) - f(a-h) &= f'(a) + f'''(\xi)\frac{h^2}{3!}.
\end{aligned}$$

For $f(x) = \ln(x)$ and $h = 0.1$ we have :

$$\begin{aligned}
f'(a) &\approx \frac{1}{0.2}(\ln(.6) - \ln(.4)) \\
&\approx 2.02733.
\end{aligned}$$

□

Formulas

1.) Divided Difference Table: The rows are number $k = 0, 1, 2, 3, \ldots$ and there is a column of the x values (column # -1 ?) after which the columns are numbered $\ell = 0, 1, 2, 3, \ldots$.

$$\begin{array}{ccccccc}
x_0 & y_0 & & & & & \\
x_1 & y_1 & a_1 & & & & \\
x_2 & y_2 & a_2 & b_2 & & & \\
x_3 & y_3 & a_3 & b_3 & c_3 & & \\
x_4 & y_4 & a_4 & b_4 & c_4 & d_4 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots
\end{array}$$

where the entries are calculated as follows:

$$\begin{array}{cccccccc}
x_0 & y_0 & & & & & \\
x_1 & y_1 & \frac{y_1-y_0}{x_1-x_0} & & & & \\
x_2 & y_2 & \frac{y_2-y_1}{x_2-x_1} & \frac{a_2-a_1}{x_2-x_0} & & & \\
x_3 & y_3 & \frac{y_3-y_2}{x_3-x_2} & \frac{a_3-a_2}{x_3-x_1} & \frac{b_3-b_2}{x_3-x_0} & & \\
x_4 & y_4 & \frac{y_4-y_3}{x_4-x_3} & \frac{a_4-a_3}{x_4-x_2} & \frac{b_4-b_3}{x_4-x_1} & \frac{c_4-c_3}{x_4-x_0} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots
\end{array}$$

2.) Polynomial Approximation Error:

Suppose $f \in C^{[n+1]}[a,b]$ and $P(x)$ is a polynomial approximation for $f(x)$ that contains the points $\{(x_i, y_i)\}_{i=0}^{n}$. Then for any x between $\min\{x_0, x_1, \ldots, x_n\}$ and $\max\{x_0, x_1, \ldots, x_n\}$ there exists a number ξ_x also between $\min\{x_0, x_1, \ldots, x_n\}$ and $\max\{x_0, x_1, \ldots, x_n\}$ so that

$$f(x) = P(x) + \frac{f^{[n+1]}(\xi_x)}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n).$$

3.) Taylor's Theorem. Suppose that $f \in C^{[n+1]}[a,b]$ then for each $x \in (a,b)$ there exists number $\xi_x \in (a,b)$ so that:

$$f(x) = f(a)+f'(a)(x-a)+f''(a)\frac{(x-a)^2}{2!}+\ldots+f^{[n]}(a)\frac{(x-a)^n}{n!}+f^{[n+1]}(\xi_x)\frac{(x-a)^{n+1}}{(n+1)!}.$$