MATH5630/6630 Dr. Smith Test 2, July 12, 2024.

Please show all your work; you may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled and will assume that the scratch work is not part of the final answer. You may use your (non-programmable) calculators.

Note: do all calculations with at least five significant digits.

Do six of the following problems. You may do all seven for extra credit.

Problem 1 Suppose that a researcher has estimated a certain quantity Q so that if $E(h)$ is the estimate then $Q = E(h) + O(h^2)$. If $E(h) = 1.732658$ and $E(\frac{h}{3}) = 1.746713$.

(a.) Derive Richardson's extrapolation formula for estimating the error for the estimate $E(\frac{h}{3})$.

Solution. We assume that $Q = E(h) + Mh^2$ for some constant M. Then

$$\begin{aligned} Q &= E(h) + Mh^2 \\ Q &= E\left(\frac{h}{3}\right) + M\frac{h^2}{9} \\ 0 &= E(h) - E\left(\frac{h}{3}\right) + \frac{8}{9}Mh^2 \\ \frac{Mh^2}{9} &= \frac{1}{8}(E\left(\frac{h}{3}\right) - E(h)). \end{aligned}$$

□

(b.) Use the formula obtained in (a.) to obtain an improved estimate $E(\frac{h}{3})$ that Richardson's extrapolation technique gives us.

Solution. Substituting the given values gives us:

$$\begin{aligned} \text{error} = \frac{Mh^2}{9} &= 0.00175687 \\ \text{improved } E\left(\frac{h}{3}\right) &= E\left(\frac{h}{3}\right) + \text{error} \\ &= 1.7484699. \end{aligned}$$

□

Problem 2. Use the trapezoid method to approximate the following integral:

$$\int_1^2 \ln(2x+1)dx$$

.

(a.) Do the calculation with $h = 0.5$.
(b.) Do the calculation with $h = 0.25$.
(c.) Use Richardson's extrapolation to estimate the error in the approximation obtained in (b.) above.

Solution. We use formula 1 from the formula sheet

$$\begin{aligned}
(a.) &= 1.37016 \\
(b.) &= 1.37429 \\
(c.) &= \text{improved } E\Big(\frac{h}{2}\Big) = E\Big(\frac{h}{2}\Big) + \frac{1}{3}\Big(E\Big(\frac{h}{2}\Big) - E(h)\Big) \\
&= 1.375667.
\end{aligned}$$

□

Problem 3. A mathematician wants to calculate the following integral:

$$\int_0^{1.5} e^{x^2} dx.$$

(a.) If he employs the trapezoidal method with an h value of 0.1, then what is an estimate of the error for the approximation.

Solution. We use The error term from formula 1 from the formula sheet

$$f''(x) = 2e^{x^2} + 4x^2 e^{x^2}.$$

It attains its maximum on $[0, 1.5]$ at $x = 1.5$; that maximum is 104.3651:

$$\begin{aligned}
\text{error} &= \frac{1.5}{12}(104.3651)(h^2) \\
&= 0.130456
\end{aligned}$$

□

(b.) If the mathematician wants to be certain that the error is less than 0.001, then what h value does he need and how many subintervals will he need to divide up the interval $[0, 1.5]$.

Solution. We need

$$\begin{aligned} \frac{1.5}{12}(104.3651)(h^2) &< .001 \\ h &< 0.0087552 \\ n &> 1.5/h = 171.33 \end{aligned}$$

□

Problem 4. Use the Euler method to estimate the solution of the following differential equation:

$$y'(t) = 2t - y^2 \quad y(0) = 2.$$

Use $h = 0.1$ and do two iterations.

Solution. The recursion formula is

$$w_{i+1} = w_i + h(2t_i - w_i^2).$$

i	h	t_i	w_i	$2t_i - w_i^2$
0	0.1	0	2	-4
1	0.1	0.1	1.6	-2.36
2	0.1	0.2	1.364	

□

Problem 5. Use the Euler method to estimate the solution of the following system of two differential equation:

$$\begin{aligned} u'(t) &= 3v(t) - u(t); \\ v'(t) &= 2u(t)v(t); \\ \begin{bmatrix} u(1) \\ v(1) \end{bmatrix} &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{aligned}$$

Use $h = 0.1$ and do two iterations.

Solution. The recursion formulas are

$$\begin{aligned} u_{i+1} &= u_i + h(3v_i - u_i) \\ v_{i+1} &= v_i + h(2u_i v_i). \end{aligned}$$

i	h	t_i	u_i	v_i
0	0.1	0	2	1
1	0.1	0.1	2.1	1.4
2	0.1	0.2	2.31	1.988

□

Problem 6. Use formula 4 to show the following:

(a.) Prove that for each positive integer n

$$\Gamma(n+1) = n!.$$

Solution. First consider the case $n = 1$. Then

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

becomes

$$\begin{aligned} \Gamma(n+1) &= \int_0^\infty t^n e^{-t}dt \\ \Gamma(2) &= \int_0^\infty te^{-t}dt \\ &= -te^{-t}|_0^\infty - \int_0^\infty e^{-t}dt \\ &= -te^{-t}|_0^\infty - e^{-t}|_0^\infty \\ &= -0 + 0 - (0-1) = 1 = 1! \end{aligned}$$

We continue by induction

$$\begin{aligned} \Gamma(n+1) &= \int_0^\infty t^n e^{-t}dt \\ &= -t^n e^{-t}|_0^\infty + \int_0^\infty nt^{n-1}e^{-t}dt \\ &= 0 + n\Gamma(n) \\ &= n(n-1)! = n!. \end{aligned}$$

where the last step follows from the induction hypothesis. □

(b.) Assume that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and use this information to obtain $\Gamma(\frac{3}{2})$ and $\Gamma(\frac{5}{2})$.

Solution. Notice that the equation derived above does not require that n be an integer, so we can use it to do these calculations.

$$\begin{aligned}
\Gamma(x+1) &= x\Gamma(x) \\
\Gamma\left(\frac{3}{2}\right) &= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \\
&= \frac{1}{2}\sqrt{\pi} \\
\Gamma\left(\frac{5}{2}\right) &= \frac{3}{2}\Gamma\left(\frac{3}{2}\right) \\
&= \frac{3}{4}\sqrt{\pi}.
\end{aligned}$$

□

Problem 7. Convert the following second order differential equation to a system of two first order differential equations:

$$y''(t) + (x^2+3)y'(t) - (t+xy)y(t) = 0; \quad y(0) = 2, y'(0) = 3.$$

Solution. Let $u(t) = y'(t)$ and $v(t) = y(t)$ then the second order differential equation becomes the following system:

$$\begin{aligned}
u' &= -(x^2+3)u + (t+xy)v; \\
v' &= u; \\
\begin{bmatrix} u(0) \\ v(0) \end{bmatrix} &= \begin{bmatrix} 3 \\ 2 \end{bmatrix}.
\end{aligned}$$

□