**MATH5630/6630 Dr. Smith Test 1, June 20, 2025.**

Make sure to show all your work. You may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work, make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled and will assume that the scratch work is not part of the final answer. You may use a scientific non-programable calculator.

Problem 1. Consider the function $f(x) = 3x^3 + 2x - 10$.

a.) Argue that the function has a root between $x = 1$ and $x = 2$.

*Solution.*

$$\begin{aligned} f(1) &= -5 \\ f(2) &= 18. \end{aligned}$$

□

b.) Use two iterations of the Bisection method to approximate the root, use $a = 1$ and $b = 2$ to start.

*Solution.*

$$\begin{aligned} f(1.5) &= 3.125 \\ f(1.25) &= -1.64063 \\ \text{root} &\approx 1.375 \end{aligned}$$

□

c.) How accurate is your approximation?

*Solution.* Accuracy $= 1.375 - 1.25 = 0.125$. □

d.) How many iterates would be needed to guarantee that your estimate is within 0.1 of the correct value? [Give the minimum number that you can prove works.]

*Solution.* We need

$$\frac{2-1}{2^n} < 0.1.$$

So $n = 4$ is the needed number of iterations. □

Problem 2. Again. consider the function $f(x) = 3x^3 + 2x - 10$.

a.) Use two iterations of the Newton's method to approximate the root; use $x_0 = 2$ to start.

*Solution.*

$$\begin{aligned} \text{first iteration} &= 1.5263 \\ \text{second iteration} &= 1.3643. \end{aligned}$$

□

b.) Is the approximation within 0.1 units, 0.01 units or 0.001 units?

*Solution.* We consider $f(1.3643 \pm 0.1)$, $f(1.3643 \pm 0.01)$, etc.

$$\begin{aligned} f(1.3643 + 0.1) &= 2.3487 \\ f(1.3643 - 0.1) &= -1.4079 \\ f(1.3643 + 0.01) &= 0.5364 \\ f(1.3643 - 0.01) &= 0.1613. \end{aligned}$$

So the second iteration is within 0.1 of the root but not within 0.01 of the root. □

Problem 3. Given the following data:

| $i$ | $x_i$ | $y_i$ |
|---|---|---|
| 0 | 1.0 | 2 |
| 1 | 1.2 | 2.5 |
| 2 | 1.4 | 2.8 |
| 3 | 1.5 | 2.7 |

Find the Lagrangian form of the polynomial of minimal degree that contains these points.

*Solution.*

$$2\frac{(x-1.2)(x-1.4)(x-1.5)}{(1-1.2)(1-1.4)(1-1.5)} + 2.5\frac{(x-1)(x-1.4)(x-1.5)}{(1.2-1)(1.2-1.4)(1.2-1.5)} +$$

$$+2.8\frac{(x-1)(x-1.2)(x-1.5)}{(1.4-1)(1.4-1.2)(1.4-1.5)}+2.7\frac{(x-1)(x-1.2)(x-1.4)}{(1.5-1)(1.5-1.2)(1.5-1.4)}.$$

I did not mark off if you did not reduce:

$$-50(x-1.2)(x-1.4)(x-1.5)+208\frac{1}{3}(x-1)(x-1.4)(x-1.5)+$$
$$-350(x-1)(x-1.2)(x-1.5)+180(x-1)(x-1.2)(x-1.4).$$

□

Problem 4. Given the following data:

| $i$ | $x_i$ | $f(x_i)$ | $f'(x_i)$ |
|---|---|---|---|
| 0 | 1.0 | 0.70 | 0.50 |
| 1 | 1.2 | 0.80 | 0.45 |

a. Give the Newton's form of Hermite polynomial of minimal degree that contains these points and with the required derivative.

b. Solve for the unknowns.

*Solution.* The Newton's form is:

$$P(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^2(x-1.2).$$

The unknowns can be solved repeatedly to obtain:

$$\begin{aligned} a_0 &= 0.7 \\ a_1 &= 0.5 \\ a_2 &= 0 \\ a_3 &= -1.25. \end{aligned}$$

□

Problem 5. A natural cubic spline on $[2,5]$ is defined so that

$$s(x)\begin{cases} s_0(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3 & 2 \le x \le 3 \\ s_1(x) = 5 + 3(x-3) + 4(x-3)^2 - 2(x-3)^3 & 3 \le x \le 5 \\ s_2(x) = \ldots. & \end{cases}$$

a.) Set up the equations among the four variables.

b.) Find the values of the unknown quantities $a, b, c$ and $d$.

*Solution.* We obtain one equation from the "natural" condition and the rest come from equating the function, the derivatives and the second derivative at the $x = 3$ value:

$$\begin{aligned} 2c &= 0 \\ a + b + d &= 5 \\ b + 3d &= 3 \\ 6d &= 8. \end{aligned}$$

So we have that $c = 0$ then solving for $d$ and working backwards we get

$$\begin{aligned} d &= \frac{4}{3} \\ b &= -1 \\ a &= 4\frac{2}{3}. \end{aligned}$$

□

Problem 6. a.) Use Taylor's theorem to derive a two different estimators for the derivative of a function so that the error is at least $O(h^2)$.

b.) Use these estimators to estimate the derivative of $\ln(2x+1)$ at $x = 0.5$, use $h = 0.1$.

c.) Which of the two is a better estimator?

*Solution.* Since the test was too long and I gave full credit if one estimator was derived I'll indicate the technique for one of them. We start by obtaining an $O(h^3)$ version of Taylor's series expansion (because we'll eventually need to divide by $h$):

$$f(x_0 + h) = f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{2} + f'''(\xi)\frac{h^3}{3!} + O(h^4).$$

Then we want to eliminate the $f''(x_0)$ term. So we calculate as follows (I'll use the "forward" estimations):

$$\begin{aligned} Af(x_0 + h) &= A\Big[f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{2} + f'''(\xi)\frac{h^3}{3!}\Big] \\ Bf(x_0 + 2h) &= B\Big[f(x_0) + 2f'(x_0)h + 2f''(x_0)h^2 + f'''(\xi)\frac{8h^3}{3!}\Big]. \end{aligned}$$

So that we want $A\frac{1}{2} + 2B = 0$, to eliminate that term, and $A = 4, B = -1$ works (any other solution will give the same resulting equation). Adding the two equations gives:

$$4f(x_0 + h) - f(x_0 + 2h) \quad = \quad 3f(x_0) + 2f'(x_0)h + 0f''(x_0)\frac{h^2}{2} + O(h^3).$$

Solving for $f'(x_0)$:

$$\begin{aligned} 2f'(x_0)h &= -3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h) + O(h^3) \\ f'(x_0) &= \frac{1}{2h}[-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)] + O(h^2). \end{aligned}$$

□

Extra Credit: Derive an $O(h^2)$ approximation for the second derivative $f''(x_0)$ in terms of $f$ values at some of the following points $x_0, x_0 \pm h, x_0 \pm 2h, \ldots$.

**Formulas**

1.) Divided Difference Table: The rows are number $k = 0, 1, 2, 3, \ldots$ and there is a column of the $x$ values after which the columns are numbered $\ell = 0, 1, 2, 3, \ldots$.

$$\begin{array}{cccccccc}
x_0 & y_0 & & & & & \\
x_1 & y_1 & a_1 & & & & \\
x_2 & y_2 & a_2 & b_2 & & & \\
x_3 & y_3 & a_3 & b_3 & c_3 & & \\
x_4 & y_4 & a_4 & b_4 & c_4 & d_4 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots
\end{array}$$

where the entries are calculated as follows:

$$\begin{array}{ccccccc}
x_0 & y_0 & & & & & \\
x_1 & y_1 & \frac{y_1-y_0}{x_1-x_0} & & & & \\
x_2 & y_2 & \frac{y_2-y_1}{x_2-x_1} & \frac{a_2-a_1}{x_2-x_0} & & & \\
x_3 & y_3 & \frac{y_3-y_2}{x_3-x_2} & \frac{a_3-a_2}{x_3-x_1} & \frac{b_3-b_2}{x_3-x_0} & & \\
x_4 & y_4 & \frac{y_4-y_3}{x_4-x_3} & \frac{a_4-a_3}{x_4-x_2} & \frac{b_4-b_3}{x_4-x_1} & \frac{c_4-c_3}{x_4-x_0} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots
\end{array}$$

2.) Polynomial Approximation Error:

Suppose $f \in C^{[n+1]}[a,b]$ and $P(x)$ is a polynomial approximation for $f(x)$ that contains the points $\{(x_i, y_i)\}_{i=0}^{n}$. Then for any $x$ between $\min\{x_0, x_1, \ldots, x_n\}$ and $\max\{x_0, x_1, \ldots, x_n\}$ there exists a number $\xi_x$ also between $\min\{x_0, x_1, \ldots, x_n\}$ and $\max\{x_0, x_1, \ldots, x_n\}$ so that

$$f(x) = P(x) + \frac{f^{[n+1]}(\xi_x)}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n).$$

3.) Taylor's Theorem. Suppose that $f \in C^{[n+1]}[a,b]$ then for each $x \in (a,b)$ there exists number $\xi_x \in (a,b)$ so that:

$$f(x) = f(a)+f'(a)(x-a)+f''(a)\frac{(x-a)^2}{2!}+\ldots+f^{[n]}(a)\frac{(x-a)^n}{n!}+f^{[n+1]}(\xi_x)\frac{(x-a)^n}{n!}.$$