MATH5630/6630 Dr. Smith Test 2, July 18, 2025.

Please show all your work; you may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled and will assume that the scratch work is not part of the final answer. You may use your (non-programmable) calculators.

Note: do all calculations with at least five significant digits.

Problem 1. Suppose that a researcher has estimated a certain quantity Q so that if $E(h)$ is the estimate then $Q = E(h) + O(h^2)$. If $E(h) = 2.94019$ and $E(\frac{h}{3}) = 2.98501$.

(a.) Derive Richardson's extrapolation formula for estimating the error for the estimate $E(\frac{h}{3})$.

Solution.

$$\begin{aligned}
Q &= E(h) + K_1 h^2 + O(h^3) \\
Q &= E\Big(\frac{h}{3}\Big) + K_1 \frac{h^2}{9} + O(h^3) \\
E(h) + K_1 h^2 &= E\Big(\frac{h}{3}\Big) + K_1 \frac{h^2}{9} + O(h^3) \\
K_1 \frac{8h^2}{9} &= E\Big(\frac{h}{3}\Big) - E(h) + O(h^3) \\
K_1 \Big(\frac{h^2}{9}\Big) &= \frac{1}{8}\Big(E\Big(\frac{h}{3}\Big) - E(h)\Big) + O(h^3) \\
\text{eror} &= \frac{1}{8}\Big(E\Big(\frac{h}{3}\Big) - E(h)\Big) + O(h^3).
\end{aligned}$$

□

(b.) Use the formula obtained in (a.) to obtain an improved estimate $E(\frac{h}{3})$ that Richardson's extrapolation technique gives us.

Solution.

$$\begin{aligned} Q &= E\Big(\frac{h}{3}\Big) + K_1\frac{h^2}{9} + O(h^3) \\ &= E\Big(\frac{h}{3}\Big) + \frac{1}{8}\Big(E\Big(\frac{h}{3}\Big) - E(h)\Big) + O(h^3) \\ &= \frac{1}{8}\Big(9E\Big(\frac{h}{3}\Big) - E(h)\Big) + O(h^3) \\ &\approx \frac{1}{8}\Big(9 \cdot 2.98501 - 2.94019\Big) = 2.9906125. \end{aligned}$$

□

Problem 2. Use the trapezoid method to approximate the following integral:

$$\int_0^1 e^{-2x} dx$$

.

(a.) Do the calculation with $h = 0.5$.

Solution.

$$\begin{aligned} \int_0^1 e^{-2x} dx &\approx (\frac{h}{2}\Big(e^0 + 2e^{-1} + e^{-2}\Big) \\ &\approx 0.46777 \end{aligned}$$

□

(b.) Do the calculation with $h = 0.25$.

Solution.

$$\begin{aligned} \int_0^1 e^{-2x} dx &\approx \frac{h}{2}\Big(e^0 + 2e^{-0.5} + 2e^{-1} + 2e^{-1.5} + e^{-2}\Big) \\ &\approx 0.44130. \end{aligned}$$

□

(c.) Use Richardson's extrapolation to estimate the error in the approximation obtained in (b.) above.

Solution.

$$\begin{aligned} \text{error} &\approx (E(h/2) - E(h))/3 \\ &\approx (0.44130 - 0.46777)/3 = -0.0088239. \end{aligned}$$

□

Problem 3. A mathematician wants to calculate the following integral:

$$\int_0^{1.2} \sin(x^2)dx.$$

(a.) If he employs the trapezoidal method with an h value of 0.1, then what is an estimate of the error for the approximation?

Solution. $f''(x) = 2\cos(x^2) - 4x^2 \sin(x^2)$, I'll accept any plausible estimate for $f''(\xi)$ that uses the maximum sine and cosine values of 1, the parabola assumes it's maximum absolute value (of 3.76 on $[0, 1.2]$ at the 1.2 end. So, I'll accept values of $3.76, 4, 6, 2 + 4 \cdot 1.2^2 = 7.76$ - certainly $|f''(\xi)| < 8$. Therefore,

$$\begin{aligned} \left|\frac{b-a}{12}h^2 f''(\xi)\right| &< \frac{1.2}{12}h^2 \cdot 6 = 0.6h^2. = 0.006 \\ \text{or } \left|\frac{b-a}{12}h^2 f''(\xi)\right| &< \frac{1.2}{12}h^2 \cdot 8 = 0.8h^2 = 0.008. \end{aligned}$$

[Note that the actual maximum value occurs at the $x = 1.2$ end with a value of 5.45 so our estimate of 6 to 8 is very good; it's not a bad idea to check both ends to get a feeling for the bounds.]

□

(b.) If the mathematician wants to be certain that the error is less than 0.001, then what is the minimal h value that he needs and how many subintervals will he need to divide up the interval $[0, 1.2]$.

Solution. I'll do two

$$\begin{aligned} 0.6h^2 &< 0.001 \\ h &< 0.0408 \\ \text{And } h &= \frac{b-a}{n} \\ n &= 29.4. \end{aligned}$$

So we need at least 30 intervals.

Or:

$$\begin{aligned} 0.8h^2 &< 0.001 \\ h &< 0.0354 \\ \text{And } h &= \frac{b-a}{n} \\ n &= 33.9. \end{aligned}$$

So, in this case, we need at least 34 intervals.

□

Problem 4. Use the Euler method to estimate the solution of the following differential equation:

$$y'(t) = y + \sin(2y) \quad y(1) = 2.5.$$

Use $h = 0.1$ and do two iterations. (I.e. calculate $y(1 + 2h)$.)

Solution.

n	h	t	y	$f(t,y)$
0	0.1	1	2.5	1.5411
1	0.1	1.1	2.6541	1.8264
2	0.1	1.2	2.8367	

Answer = 2.8367.

□

Problem 5. Use the Euler method to estimate the solution of the following differential equation:

$$y'(t) = y + e^t \quad y(1) = 2.$$

Use $h = 0.1$ and do two iterations. (I.e. estimate $y(1 + 2h)$.)

Solution.

n	h	t	y	$f(t,y)$
0	0.1	1	2	4.71828
1	0.1	1.1	2.4718	5.47599
2	0.1	1.2	3.0194	

Answer = 3.0194.

□

Problem 6. Suppose that the trapezoid method is used to calculate the following integral:

$$\int_0^2 e^{-3t} dt.$$

(a.) Estimate the error that the error formula gives for $h = 0.1$.

Solution. The second derivative is $f''(x) = 9e^{-3x}$ which attains its maximum over the interval $[0, 2]$ at $x = 0$; the maximum is 9.

$$\begin{aligned} |\text{error}| &= \frac{b-a}{12} h^2 f''(\xi) \\ &< \frac{2}{12} h^2 9 \\ &< 1.5h^2 \\ &< 0.015. \end{aligned}$$

□

(b.) What value of h should be used to guarantee that the absolute value of the error is less than 0.0001?

Solution. From part (a)

$$\begin{aligned} |\text{error}| &< 1.5h^2 \\ 1.5h^2 &< 0.0001 \\ h &< \sqrt{\frac{.0001}{1.5}} = 0.008165. \end{aligned}$$

So $h < 0.008166$. □

Formulas List.

1.) Error term for the trapezoid method: $T(h) \approx \int_a^b f(x)dx$

$$\text{error} = -\frac{b-a}{12}h^2 f''(\xi)$$

for some $\xi \in (a, b)$.

Take Home Portion.

You may use your textbook and notes. By handing in your work, you affirm that the work is entirely your own and that you have not used any outside help besides your textbook and notes. You may also use MatLab or equivalent software.

The work is to be turned in at class time: 9:45 Monday July 21.

Problem 7. Consider the following integral

$$\int_0^4 e^{-\frac{x}{2}}\,dx.$$

Let $T(h)$ denote the trapezoid approximation of the integral in terms of the h-value. Calculate $T(0.1), T(0.05)$ and $T(0.025)$; then use these figures to calculate a Romberg type estimate of the integral that would be $O(h^6)$.

Solution.

$$\begin{aligned}
T(h) &= \frac{h}{2}\Big(1 + 2e^{-.05} + 2e^{-.1} + \ldots + 2e^{-1.95} + e^{-2}\Big)\\
T(0.1) &= 1.729689695\\
T(0.05) &= 1.729419502\\
T(0.025) &= 1.729351951
\end{aligned}$$

Let us define $R(h)$ as follows

$$R(h) = \frac{4T(h/2) - T(h)}{3}.$$

Then we want $(16R(h/2) - R(h))/15$:

$$\begin{aligned}
R(0.1) &= 1.729329437\\
R(0.05) &= 1.729329434\\
\text{answer} &= 1.7293294338
\end{aligned}$$

Note that the exact answer is $2 - 2e^{-2} = 1.729329433527$. □

Problem 8. Estimate (within .01) the value of r so that

$$\int_0^r e^{x^3} \quad = \quad 2.$$

Then show how you obtained the value within this error.

Solution. since I had just set up the romberg integration on my computer, I used that to do the integration Then I used the bisection method since it gave me a clear error term. I started with a guess of around 1.5.

$$r = \quad 1.18125 \pm 0.00625.$$

Continuing, I arrived at the following closer answer [which I believe is correct to five decimal places]:

$$r = \quad 1.17678.$$

□