Topology 7500, Fall 2023
Dr. Michel Smith
Hand In Project 1

Problems 2 and 3 are due Monday Sept 18. Problem 1 will be due sometime after a proof of the compactness of an interval in the reals has been presented - students are allowed to look up their old proofs (e.g. from analysis) for this theorem and they may collaborate on this proof for $\mathbb{R}$.

Problem 1. Let X be an ordered topological space with respect to the order $<$.

Definitions: If $M \subset X$, then the point $u \in X$ is called an *upper bound* for M iff for all $x \in M$ we have $x \leq u$. If ℓ is an upper bound for the set M and whenever $x < \ell$ then x is not an upper bound for M then ℓ is called a *least upper bound* for M. The ordered space X is said to satisfy the *least upper bound* property if and only if every subset of M has a least upper bound.

Prove the following:

Theorem P01.1. If X is an ordered topological space which has a first and last point, and X has the least upper bound property, then X is compact.

Theorem P01.1. If X is a well-ordered topological space which has a last point, then X is compact.

Problem 2. Let X denote the real numbers with the following basis $\mathcal{B}$ for the topology: If x is irrational and $\epsilon > 0$ then $B_\epsilon(x) = \{t| \; |t - x| < \epsilon\}$ is in $\mathcal{B}$; if x is rational then $\{x\}$ is in $\mathcal{B}$.

(a.) Show that X is first countable.

(b.) Determine whether or not X is regular.

[Hint: show that $\mathcal{T}(\mathbb{R})$, the standard topology on the reals is a subset of $\mathcal{T}(X)$ the topology of X.]

Problem 3. Let X be the upper half plane: $X = \{(x, y) \in \mathbb{E}^2| \; y \geq 0\}$. Define a basis for the topology as follows. If $P = (p, q)$ and $q > 0$ then if $\epsilon > 0$ then the set $\{(x, y)| \sqrt{(x-p)^2 + (y-q)^2} < \epsilon\} \cap \{(x, y)|y > 0\}$ is a basis element; if $P = (p, q)$ and $q = 0$ then the set $\{(x, y)| \sqrt{(x - p)^2 + (y - \epsilon)^2} < \epsilon\} \cup \{(p, 0)\}$ is a basis element. This is called the "tangent disc" space.

(a.) Show that X is Hausdorff.

(b.) Show that X is first countable.

(c.) Determine whether or not X is regular.

[Note: you may use anything you know about plane geometry to do this.]