Topology Notes 01.
Basic Axioms and Definitions.

A *topological space* is a pair $(X, \mathcal{T})$ where X consists of a set of objects called *points* and $\mathcal{T}$ is a collection of subsets of X called *open sets* such that the following are satisfied:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
2. If $A \in \mathcal{T}$ and $B \in \mathcal{T}$ then $A \cap B \in \mathcal{T}$.
3. If $\mathcal{U} \subset \mathcal{T}$ then $\cup\{u|u \in \mathcal{U}\} \in \mathcal{T}$.

The collection $\mathcal{T}$ is called the topology of X. The statement "$(X, \mathcal{T})$ is a topological space" is generally abbreviated by "X is a (topological) space" when the associated topology is otherwise understood.

Axiom T_0. If p and q are two points of of the space X then there is an open set that contains one of these points and not the other.

Axiom T_1. If p and q are two points of the space X then there is an open set that contains p and not q.

Axiom T_2. If p and q are two points of the space X then there exist disjoint open sets A and B containing p and q respectively. A topological space that satisfies Axiom T_2 is called a Hausdorff space.

Exercise 1.1. Determine the implications among these three axioms. In other words, determine which axiom does or does not imply the other. Find examples for the case where axiom i does not imply axiom j.

Exercise 1.2. Let X be a point set.
Let $\mathcal{T}_1 = \{\emptyset, X\}$.
Let $\mathcal{T}_2 = \{U|U \subset X\}$.
Show that $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are topological spaces. Determine if these spaces are Hausdorff.

Definition. If X is a set and $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies for X and $\mathcal{T}_1 \subsetneq \mathcal{T}_2$ then the topology $\mathcal{T}_1$ is said to be a *courser* topology than $\mathcal{T}_2$ and $\mathcal{T}_2$ is said to be a *finer* topology than $\mathcal{T}_1$

Technique 1.1: Given a set of points X one way to potentially produce a topology on X is to define a collection of subsets $\mathcal{B}$ of X that has the property that if $A, B \in \mathcal{B}$ and $x \in A \cap B$ then there is an element C of $\mathcal{B}$ so that $x \in C \subset A \cap B$; and define $\mathcal{T}(X)$ by $U \in \mathcal{T}(X)$ if and only if there is a subcollection $G \subset \mathcal{T}(X)$ so that $U = \cup\{g | g \in G\}$.

Question: Given a set X will technique 1.1 always produce a topology for a space X.

Example 1.1 The Reals.

(a.) Let $\mathbb{R}$ denote the real numbers. Then define $U \in \mathcal{T}$ if and only if for each point $u \in U$ there are two numbers a and b with $a < u < b$ so that the $(a, b) = \{x | a < x < b\} \subset U$. [Note: $(a, b) = \{x | a < x < b\}$ is called a segment.]

(b.) Let $\mathcal{B}$ be the set of all segments of the real numbers with rational endpoints, then does technique 1.1 produce the same topology for the reals as in part (a)?

Example 1.2: The Sorgenfrey Line.

(a.) Let $\mathbb{R}$ denote the real numbers. Then define $U \in \mathcal{T}$ if and only if for each point $u \in U$ there are two numbers a and b with $a \leq u < b$ so that $[a, b) = \{x | a \leq x < b\} \subset U$.

(b.) Let $\mathcal{B}$ be the set of all sets of the form $[a, b)$ with with rational endpoints a and b, then does technique 1.1 produce the same topology for the reals as in part (a)?

Example 1.3: The plane E^2.

(a.) Let $E^2 = \{(x, y) | x, y \in \mathbb{R}\}$. Then define $U \in \mathcal{T}$ if and only if for each point $u = (p, q) \in U$ there is a number $r > 0$ so that $B_r(u) = \{(x, y) | \sqrt{(x-p)^2 + (y-q)^2} < r\} \subset U$.

(b.) Let $\mathcal{B} = \{B_\epsilon(P) | P \in E^2, \epsilon > 0 \text{ and } \epsilon \text{ is rational }\}$. Then does technique 1.1 produce the same topology for the reals as in part (a)?

Example 1.4: The Taxicab plane E^2. Let $E^2 = \{(x, y) | x, y \in \mathbb{R}\}$. Then define $U \in \mathcal{T}$ if and only if for each point $u = (p, q) \in U$ there is a number $r > 0$ so that $B^T_r(u) = \{(x, y) |\ |x - p| + |y - q| < r\} \subset U$.

Exercise 1.3. For each of the above examples, show that collection $\mathcal{T}$ is a topology and that the space satisfies axioms $\mathrm{T}_0, \mathrm{T}_1$, and T_2.

Example 1.5. Another topology on the Reals. Let $\mathbb{R}$ denote the real numbers. For each $p \in \mathbb{R}$ and $\epsilon > 0$ define $R_\epsilon(p) = \{x \in \mathbb{R} |\ |x - p| < \epsilon$ if x is irrational and $|x - p| < 2\epsilon$ if x is rational$\}$. Let $U \in \mathcal{T}$ if and only if for each $u \in U$ there is a number ϵ so that $R_\epsilon(u) \subset U$.

Exercise 1.3e. Is $\mathcal{T}$ of example 1.5 a topology for the Reals? If so what is its relation to the topologies of examples 1.1 and 1.2?

Example 1.6. (The co-finite topology). Let X be an arbitrary set. Define the set $U \subset X$ to be open if and only if $X - U$ is finite.

Exercise. For X equal the reals, determine the subset relation (courser and finer topologies) among the topologies defined on the reals above. (I.e. for the topologies defined by Exercise 1.2, examples 1.1, 1.2, 1.5 and 1.6.)

Definition. If $(X, \mathcal{T})$ is a topological space and $M \subset X$ is a point set then the point p is said to be a *limit point* of the set M if and only if each open set containing p contains a point of M distinct from p.

For the following theorems assume that all spaces are Hausdorff; determine if the theorems are true for T_0 or T_1 spaces (I think some are not.)

Theorem 1.1. If X is a finite Hausdorff space then no point of X is a limit point of X.

Definition. Suppose $(X, \mathcal{T})$ is a topological space and $M \subset X$. Then the *derived set* of M denoted by M' is the set of limit points of M. The *closure* of M denoted by $\overline{M}$ is the set $M \cup M'$.

Theorem 1.2. If $(X, \mathcal{T})$ is a topological space and $M \subset X$ then $\overline{M} = \overline{\overline{M}}$.

Definition. The set $M \subset X$ is said to be *closed* if and only if every limit point of M is in M.

Theorem 1.3. Suppose $(X, \mathcal{T})$ is a topological space and $M \subset X$. Then M is closed if and only if $X - M$ is open.

Theorem 1.4. Suppose $(X, \mathcal{T})$ is a topological space and $M \subset X$. Then M' is closed.

Theorem 1.5. If p is a limit point of the set M, then every open set containing p contains infinitely many points of M.

Theorem 1.6. If each of A and B is a closed subset of the space X, then $A \cap B$ and $A \cup B$ are closed.

Question: Does Theorem 1.6 hold if X is not required to be Hausdorff? Is every space satisfying the conditions of Theorem 1.6 for arbitrary pairs of closed sets a Hausdorff space?

Definition. Suppose $(X, \mathcal{T})$ is a topological space and $M \subset X$. Then the point p is called a *boundary point* of M iff every open set containing p contains a point in M and a point not in M. We denote the boundary of the set M by $Bd_X(M)$. Note that the subscript may be omitted if the underlying space is understood.

Exercise 1.4.

a. Find an example of a set that has no boundary point.

b. Find an example of a set every point of which is a boundary point.

c & d. Repeat a and b above with the added requirement that the set be open (c) or closed (d).

[Note that as part of this exercise you need to define a space X and a topology on that space, and then find a set in that space that has the required property.]

Theorem 1.7. If X is a topological space and $M \subset X$, then $Bd(M) = \overline{M} \cap \overline{(X - M)}$.

Corollary 1.7. If X is a topological space and $M \subset X$, then $Bd(M)$ is closed.

Definition. Suppose $(X, \mathcal{T})$ is a topological space and $M \subset X$. Then the *interior* of M, denoted by $Int(M)$ is the set to which x belongs if and only if there is an open set containing x lying in M.

Exercise 1.5. In all the above theorems determine whether or not the theorem holds under the weaker T_0 or T_1 axioms.

Exercise 1.6. Suppose X is a topological space and $M \subset X$. Prove the following. [In each case determine the weakest axiom that is needed to prove the statement. Caution: at least one of these is false! Note that I may not warn you in the future. In all the cases where the statement is false you should provide a counter example.]

a. $Int(M)$ is open.
b. $Int(Bd(M)) = \emptyset$.
c. $Int(Int(M)) = Int(M)$.
d. $Bd(Bd(M)) = Bd(M)$.
e. $Int(A \cap B) = Int(A) \cap Int(B)$.
f. $Bd(A \cap B) = Bd(A) \cap Bd(B)$.
g. $Int(M) \cap Int(X - M) = \emptyset$.
h. $(X - Int(M)) - Int(X - M) = Bd(M)$.
i. M is open iff $M \cap Bd(M) = \emptyset$.
j. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
k. $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
l. U is open iff $U = Int(U)$.
m. $X - Int(A) = \overline{X - A}$.
n. $X = Int(M) \cup Bd(M) \cup Int(X - M)$.
o. A is closed iff $Bd(A) \subset A$.