Topology Notes 02
Basic Axioms

Definition.

A space X is said to be *regular* if and only if for each point $x \in X$ and each closed set $H \subset X$ not containing x there exists disjoint open sets U and V such that $x \in U$ and $H \subset V$.

Axiom T_3. The space X is a regular Hausdorff space.

Question. If X is regular then does it follow that it is Hausdorff. (I.e. is the condition of Hausdorff above redundant?)

Definition.

A space X is said to be *normal* if and only if for each pair of disjoint closed sets H and K there exists a pair U and V of disjoint open sets such that $H \subset U$ and $K \subset V$.

Axiom T_4. The space X is a normal T_3 space.

Question. If X is normal then does it follow that it is regular. (I.e. is the condition of T_3 above redundant?)

Exercise 2.1. (Continuation from 01 Notes.)

Determine for which pairs i and j, it is true that if X satisfies Axiom T_i then it also satisfies T_j.

Examples: Consider the examples in the 01 notes. Determine if these examples are regular and if they are normal.

Definition.

Suppose that $(X, \mathcal{T})$ is a topological space. A *basis* $\mathcal{B}$ for the topology of X is a subset of $\mathcal{T}$ such that if $x \in X$ and U is an open set containing x then there exists an element R in $\mathcal{B}$ containing x and lying in U.

Theorem 2.1. Suppose that $(X, \mathcal{T})$ is a topological space and $\mathcal{B}$ is a basis for the space. Then $\mathcal{T} = \{\cup u | u \subset \mathcal{B}\}$.

Theorem 2.2. Suppose that $(X, \mathcal{T})$ is a topological space, $\mathcal{B}$ is a basis for the space and $M \subset X$. Then the point p is a limit point of M iff every element of the basis $\mathcal{B}$ containing p contains a point of M distinct from p.

Theorem 2.3. Suppose that $\mathcal{B}$ is a collection of subsets of X so that:

1. Every point of X is in some element of $\mathcal{B}$.
2. If $p \in X$ and A and B are elements of $\mathcal{B}$ both containing p, then there is an element of $\mathcal{B}$ containing p lying in $A \cap B$.

Then $\mathcal{T} = \{\cup W | W \subset \mathcal{B}\}$ is a topology for X.

Definition: Under this hypothesis, the topology $\mathcal{T}$ is said to be generated by the basis $\mathcal{B}$.

Notational convention. Sometimes the expression $cl(M)$ is used to to denote $\overline{M}$ the closure of M.

Definition.

Let X be a topological space. The space X is called a *Moore space* if and only if there is a sequence $G_1, G_2, G_3, ...$ such that:

1. for each positive integer n, G_n is a basis for the topology of X;
2. for each positive integer n, $G_{n+1} \subset G_n$; and
3. if R is an open set, $A \in R$ and $B \in R$, then there is a positive integer m such that if g is an element of G_m containing A then $cl(g)$ is a subset of R and if A is distinct from B then $cl(g)$ does not contain B.

Theorem 2.4. If X is a Moore space then X satisfies axioms T_0, T_1, T_2 and T_3.

Definition. Suppose that X is a set and $d : X \times X \to [0, \infty)$ so that:

(1.) If $x, y \in X$ then $d(x, y) = 0$ if and only if $x = y$.

(2.) If $x, y \in X$ then $d(x, y) = d(y, x)$.

(3.) If $x, y, z \in X$ then $d(x, z) \leq d(x, y) + d(y, z)$.

Then d is called a *metric* for X.

Notation. If d is a metric for the space X, $x \in X$ and $\epsilon \in \mathbb{R}$ then:

$$B_\epsilon(x) = \{t \in X | d(x, t) < \epsilon\}$$

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Definition. The topological space X is said to be a metric space if and only if there is a metric d so that collection if $\mathcal{B} = \{B_\epsilon(x) | x \in X, \epsilon \in \mathbb{R}\}$ forms a basis that generates the topology of X.

Exercise 2.2. Determine the relationships between X being regular, normal, a Moore space and a metric space.

Theorem 2.5. If X is a Moore space and $p \in X$ then there exists an infinite sequence $R_1, R_2, R_3, ...$ such that:

1. for each n, R_n is an open set;
2. $\{p\} = \bigcap_{i=1}^{\infty} R_i$;
3. for every positive integer n, $cl(R_{n+1}) \subset R_n$;
4. if R is an open set containing p then there exists an integer n so that $cl(R_n) \subset R$.

Definition. Suppose that $(X, \mathcal{T})$ is a topological space and $x \in X$. Then $\mathcal{B}$ is said to be a *local basis* at x if and only if every element of $\mathcal{B}$ is an open set containing x and if R is an open set containing x then there is an element of $\mathcal{B}$ containing x and lying in R.

Definition. The space X is said to be *first countable* provided there is a local basis at each point of X that is countable.

Corollary 2.5. If X is a Moore space then X is first countable.

Theorem 2.6. Suppose that $(X, \mathcal{T})$ is a topological space and $S \subset X$ and $\mathcal{T}(S) = \{R \cap S | R \in \mathcal{T}\}$. Then $(S, \mathcal{T}(S))$ is a topological space.

Definition.

Under the hypothesis of Theorem 2.6 the space $(S, \mathcal{T}(S))$ is called a *subspace* of X with the subspace topology.

Exercise 2.3. For each of the properties: $\mathrm{T}_i, i = 0, ...4$, being a Moore space, being first countable, determine if it is true that if a space X has that property then every subspace of X also has that property.

Exercise 2.4. Show that a metric space is a Moore space.