Topology Notes 03
Order Relations, Compactness

Definition. Suppose that M is a set and "$<$" is a relation on M so that if each of x, y and z is an element of M then:

1. If $x \neq y$ then either $x < y$ or $y < x$;
2. If $x < y$ then $y \not< x$;
3. If $x < y$ and $y < z$ then $x < z$.

Then M is said to be ordered with respect to the relation "$<$".

Exercise 3.1. Suppose that M is a set and M is ordered with respect to the relation "$<$". Let $\mathcal{B}_<$ be the set to which U belongs if and only if:

1. there are points x and y so that $U = \{p \in M | x < p < y\}$,
2. there is a point t so that $U = \{p \in M | p < t\}$, or
3. there is a point t so that $U = \{p \in M | t < p\}$.

Then:

i. Show that $\mathcal{B}_<$ satisfies the hypothesis of Theorem 2.3. The topology generated by $\mathcal{B}_<$ is called the order topology on M.

Determine which topological properties are satisfied by by the order topology.

ii. For which n is M a T_n space.

iii. Is M necessarily a first countable space?

iv. Is M necessarily a Moore space?

Definition. Suppose that X is a topological space and that G is a collection of subsets of X. Then the collection G is said to *cover* the subset M of X if and only if each point of M lies in some element of G.

Definition. Suppose that X is a topological space and $M \subset X$. Then M is said to be compact provided it is true that if G is a collection of open sets that covers M then some finite subcollection covers M.

Recall that we are assuming that all spaces are Hausdorff (T_2)

Theorem 3.1. If M is a compact subset of the space X then M is closed.

Theorem 3.2. If M is a compact subset of the space X and H is a closed subset of M, then H is compact.

Theorem 3.3. Every infinite subset of a compact set has a limit point.

Definition. Suppose that M is ordered with respect to the relation "$<$" and $H \subset M$. Then p is said to be the *first element* of H if and only if $p \in H$ and $p = x$ or $p < x$ for every $x \in H$.

Definition. Suppose that M is ordered with respect to the order relation "$<$". Then M is said to be *well ordered* with respect to "$<$" if and only if every subset of M has a first element.

Axiom of Choice. Suppose that G is a collection of sets. Then there is a function $F : G \to \cup G$ so that $F(g) \in g$ for all $g \in G$.

Well ordering Theorem (Axiom of Choice.) If M is a set then there exists an order relation on M so that M is well ordered with respect to that relation.

Exercise 3.2. Show that if M is well ordered, then M satisfies the Axiom of Choice.

Exercise 3.3. Show that there exists a well ordering for the set of rational numbers.

Definition. Suppose that M and N are two sets with orderings $<_M$ and $<_N$ respectively. Then the function $\phi : M \to N$ is said to be an *order isomorphism* provided that $\phi(x) <_N \phi(y)$ if and only if $x <_M y$ for all $x, y \in M$. If the function is onto then the sets M and N are said to be order isomorphic (with respect to their respective orderings.)

Definition. If M is well ordered with respect to $<$ then $I \subset M$ is called an *initial segment* of M if and only if there is a point $t \in M$ so that $I = \{x | x < t\}$.

Theorem 3.4. If M and N are two well ordered sets then either they are order isomorphic or one is order isomorphic to an initial segment of the other. Furthermore this order isomorphism is unique.

Theorem 3.5. Suppose that M is well ordered and M has a last element. Suppose further that M is given the order topology, then M is compact.

Definition. The collection G of sets is said to be *monotonic* if and only if for each pair of sets H and K in G, either $H \subset K$ or $K \subset H$.

Definition. The point set M is said to be *perfectly compact* if and only if it is true that if G is a monotonic collection of non-empty subsets of M then there is a point p that is a point or a limit point of every every element of G.

Theorem 3.6. If M is a compact subset of a Moore space then M is perfectly compact.

Lemma. If M is a closed and perfectly compact subset of a Moore space and G is a collection of open sets covering M so that G is well ordered with respect to some order relation "$<$", then there exists an element $g \in G$ such that if J is the collection consisting of g together with the elements of G that precede g, then J covers M.

Lemma 2. If M is a closed perfectly compact subset of a metric space X and X has a countable basis (is completely separable), then M is compact.

Theorem 3.7. If M is a closed perfectly compact subset of a Moore space, then M is compact.

Definition. If $\{p_i\}_{i=1}^{\infty}$ is a sequence of points of the space S, then p is the *sequential limit point* of the sequence means that for each open set R containing p there exists an integer N so that $p_n \in R$ for all $n > N$.

Theorem 3.8. Suppose that S is a first countable space, $M \subset S$ and p is a limit point of M. Then there exists an infinite sequence of point of M $\{p_i\}_{i=1}^{\infty}$ so that p is the sequential limit point of that sequence.

Theorem 3.9. Suppose that S is a Moore space and $M = \{p_i\}_{i=1}^{\infty} \subset S$ is a set having the property that every infinite subset has a limit point. Then some subsequence of the sequence $\{p_i\}_{i=1}^{\infty}$ that has a sequential limit point.

Exercise 3.4. Show that Theorem 3.9 does not hold for a Hausdorff space.

Exercise 3.5. Determine if the following is a theorem: If M is a Hausdorff space that has the property that every infinite subset of M has a limit point, then M is compact.

Definition. Suppose that each of X and Y is a topological space and $f : X \to Y$ is a function. Then f is said to be continuous if and only if for each $x \in X$ and each open set $U \subset Y$ containing $f(x)$ there exists an open set $R \subset X$ containing x so that $f(x) \in U$ for all $x \in R$. [Equivalently: $f(R) \subset U$ where $f(R) = \{y|\exists x \in X \ni y = f(x)\}$].

Theorem 3.10. Suppose that each of X and Y is a topological space and $f : X \to Y$ is an onto function. Then f is continuous if and only if for each open set $U \subset Y$, $f^{-1}(U)$ is open. [Where $f^{-1}(U) = \{x \in X|f(x) \in U\}$.]

Theorem 3.11. The topological space X is normal if and only if for each pair of disjoint closed subsets H and K of X, there exists a continuous function $f : X \to [0, 1]$ such that $f(H) = 0$ and $f(K) = 1$.

Exercise 3.6. Suppose that each of X and Y is a topological space and $f : X \to Y$ is a continuous onto function.

I. Determine if it is true that if X is has property $\mathcal{P}$ then Y must have property $\mathcal{P}$ where property $\mathcal{P}$ is:

a. Hausdorff.
b. Regular.
c. Normal.
d. A Moore space.
e. First countable.
f. Compact.

II. Determine if it is true that if Y is has property $\mathcal{P}$ then X must have property $\mathcal{P}$ where property $\mathcal{P}$ is as above.

Exercise 3.7 and 3.8. Repeat exercise 3.6 but suppose that each of X and Y is a topological space and $f : X \to Y$ is a continuous 1-1 onto function; repeat the exercise on the supposition that f is a continuous 1-1 onto function and that f^{-1} is continuous. [It would be helpful to produce a continuous 1-1 onto function whose inverse is not continuous.]