Topology Notes 04
Metric Spaces

Definition. Suppose that X is a topology space; then the function $d : X \times X \to \mathbb{R}$ is called a *metric* provided that d satisfies the following conditions: if each of x, y and z is a point of X then:

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ only in case that $x = y$;
2. $d(x, y) = d(y, x)$; and
3. $d(x, y) + d(y, z) \geq d(x, z)$.

Definition. Suppose that X is a topological space and the function d is a metric function; if $p \in X$ and $\epsilon > 0$ then $B_\epsilon(p) = \{x \in X | d(x, p) < \epsilon\}$ is called the ϵ-ball centered at p. Then the space X said to be a *metric space* if and only if the collection $\mathcal{B} = \{B_\epsilon(p) | p \in X, \ \epsilon > 0\}$ is a basis for the topology of X.

Theorem 4.1. If X is a metric space then X is a Moore space.

Corollary. If X is a metric space then X is first countable at each of its points.

Theorem 4.2. If X is a metric space then X is normal.

Definition. Suppose that X is a topological space then the subset M of X is said to be *dense* in X if and only if each non-empty open set contains a point of M.

Exercise. Verify that $M \subset X$ is dense in X if and only if $\overline{M} = X$.

Definition. The space X is said to be *separable* if and only if there is countable subset of X that is dense in X.

Theorem 4.3. A compact Moore space is separable.

Definition. The space X is said to be *completely separable* (or *second countable*) if it has a countable basis.

Theorem 4.4. A compact Moore space is completely separable.

Exercise. Does separability imply complete separability? What about the converse?

Exercise.

a.) Find an example of a compact Hausdorff space that is not separable.

b.) Find an example of a compact Hausdorff space that is not completely separable.

Definition. Suppose that X is a topological space and $M \subset X$. Then M is said to be *nowhere dense* in X if and only if every open set contains an open set that does not intersect M.

Exercise. Show that M is nowhere dense in X if and only if $X - \overline{M}$ is a dense open subset of X.

Theorem 4.5. No compact Hausdorff space is the union of countable many nowhere dense subsets.

Corollary. If X is a compact Hausdorff space and every point of X is a limit point of X, then X is uncountable.

Definition. If p is a point of the topological space X and $U \subset X$, then U is said to be a *neighborhood* of p if and only if $p \in Int(U)$.

Definition. The space X is said to be locally compact if and only if there is a compact neighborhood of each point of X.

Theorem 4.6. No locally compact Hausdorff space X is the union of countably many closed subsets such that if g is any one of them then every point of g is a limit point of $X - g$.

Exercise. Suppose that X is a metric space with metric d. Suppose that $\rho : X \times X \to \mathbb{R}$ is defined by:

$\rho(x, y) = d(x, y)$ if $d(x, y) \leq 1$,

$\rho(x, y) = 1$ if $d(x, y) > 1$.

Show that ρ is a metric for X and generates the same topology as d.

Definition. Suppose that X is a metric space with metric d. Then the sequence $\{p_i\}_{i=1}^{\infty}$ is said to be a *Cauchy sequence* if and only if for each $\epsilon > 0$ there exists an integer N so that if n and m are integers larger than N then $d(x_n, x_m) < \epsilon$.

Exercise. Suppose that X is a metric space with metric d and $\{p_i\}_{i=1}^{\infty}$ is a Cauchy sequence. Show that the set $\{x | x = p_i$ for some i $\}$ has at most one limit point.

Definition. Suppose that X is a metric space with metric d. Then the metric is said to be a *complete metric* for X if and only if every Cauchy sequence (with respect to the metric d) in X converges. A space is said to be a *complete metric space* if and only if there is a complete metric for the topology of X.

Definition. Suppose that X is a metric space with metric d and $H \subset X$. Then the *diameter* of H with respect to d is the number $diam(H) = lub\{d(x, y) | x, y \in H\}$ provided that least upper bound exists.

Theorem 4.7. Suppose that X is a metric space with metric d. Then the following are equivalent.

a. The metric d is a complete metric;

b. If $\{M_i\}_{i=1}^{\infty}$ is a monotonic sequence of non-empty closed subsets of X such that the diameters of the sets limits to 0, then there is a point common to all the elements of the sequence.

Definition. Suppose that X is a topological space and that $M \subset X$. The set M is said to be of the *first category* or a *meager* set if it is the union of a countable collection of open sets. A set which is not of the first category is said to be of the *second category* or *nonmeager*.

Definition. Suppose that X is a topological space and that $M \subset X$. Then M is said to be a G_δ *set* if it is the common part of a countable collection of open sets; M is said to be an F_σ *set* if it is the union of a countable collection of closed sets.

Exercise. Suppose that X is a topological space and that $M \subset X$. The set M is a G_δ if and only if $X - M$ is an F_σ set.

Definition. A Baire space is a topological space in which the common part of every countable collection of dense open sets is non-empty.

Exercise. Show that if X is a Baire space then the common part of a countable collection of dense open sets in X is dense in X.

Theorem 4.8 [Baire Category Theorem]. If X is a complete metric space then X is a Baire space.

Definition. The space X is said to be locally compact if and only if for each $x \in X$ there is an open set U containing x so that the closure of U is compact.

Theorem 4.8B [Baire Category Theorem for Hausdorff spaces]. If X is a locally compact Hausdorff space then X is a Baire space.

Theorem 4.9. Suppose that X is a complete metric space. Then the subset M of X is a complete metric space if and only if M is a G_δ set in X.

Definition. Suppose that M is a subset of the space X and $f : M \to Y$ is a function. Then $F : M \to Y$ is said to be an extension of f if and only if F is a function from X to Y so that $F(x) = f(x)$ for all points of x in M.

Theorem 4.10 [Tietze Extension Theorem.] Suppose that X is a normal space and M is a closed subspace of X. Then any continuous function from M into the interval $[0, 1]$ (or into the reals) can be extended to a continuous function from X into $[0, 1]$ (or into the reals).

Theorem 4.11 If S is a separable metric space then it is completely separable.