Product Spaces.

Definition. Suppose that X and Y are topological spaces with topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$. Then the *topological product* space of X and Y denoted by $X \times Y$ is defined as follows:

1. the points of $X \times Y$ are the elements of the cartesian product of the spaces: $X \times Y = \{(x, y) | x \in X,\ y \in Y\}$;

2. the collection $B = \{U \times V | U \in \mathcal{T}_X,\ V \in \mathcal{T}_Y\}$ forms a basis for the topology $\mathcal{T}_{X \times Y}$.

Exercise 5.1. Show that the topology for $X \times Y$ is well defined; i.e. show that the set B of the definition above satisfies the hypothesis of theorem 2.3.

Exercise 5.2. Determine for which n it is true that if each of X and Y is a T_n space then so is $X \times Y$.

Exercise 5.3. Determine for which properties it is true that if X and Y have the property then so does $X \times Y$. Consider in particular the following properties:

a. Hausdorff,
b. compact,
c. separable,
d. first countable,
e. completely separable,
f. metric.

Definition. Suppose that for some index set I, $\{(X_i, \mathcal{T}_{X_i})\}_{i \in I}$ is a collection of topological spaces. Then the topological product space $\Pi_{i \in I} X_i$ is defined as follows:

1. $\Pi_{i \in I} X_i = \{p | p = \{p_i\}_{i \in I}, p_i \in X_i,\ \text{for all } i \in I\}$;

2. the collection $B = \{\Pi_{i \in I} U_i | U_i \in \mathcal{T}_{X_i},\ U_i = X_i \text{ for all but finitely many } i \in I\}$ forms a basis for the topology.

Exercise 5.4. Show that the topology for the topological product space $X = \Pi_{i \in I} X_i$ is well defined.

Repeat the other above exercises for the topological product spaces.

Definition. The collection G of set is said to have the *finite intersection property* if and only if for each finite collection of element from G $\{g_i\}_{i=1}^N$ we have $\cap_{i=1}^N g_i \neq \emptyset$.

Theorem 5.1 A. The space X is compact if and only if for each collection G of non-empty closed sets with the finite property, $\cap\{g|g \in G\} \neq \emptyset$.

Lemma. If each of X and Y is compact then so is $X \times Y$.

Theorem 5.1 B. Suppose that I is an index set and for each $i \in I$, X_i is a compact space. Then $\Pi_{i\in I} X_i$ is compact.

Definition. Suppose that for some index set I, $\{(X_i, \mathcal{T}_{X_i})\}_{i\in I}$ is a collection of topological spaces. Then the *topological box product* space $\Box_{i\in I} X_i$ is defined as follows:

1. $\Box_{i\in I} X_i = \{p|p = \{p_i\}_{i\in I}, p_i \in X_i, \text{ for all } i \in I\}$;
2. the collection $B = \{\Pi_{i\in I} U_i | U_i \in \mathcal{T}_{X_i}, \text{ for all } i \in I\}$ forms a basis for the topology.

Do the usual exercises.

Exercise 5.5. Show that the countable box product $\Box_{i=1}^{\infty}[0, 1]$ of copies of the unit interval $[0, 1]$ with the usual topology in the Reals is not compact.

Exercise 5.6. Let I be an index set and for each $i \in I$ let X_i be a topological space. Let $F : \Box_{i\in I} X_i \to \Pi_{i\in I} X_i$ be defined by $F(x) = x$. Show that F is continuous. Give an example to show that F^{-1} is not necessarily continuous. Under what conditions can you be certain that F^{-1} is continuous.

Notation. If $X = \Pi_{i\in I} X_i$ then the projection map π_i onto the i^{th} coordinate is the function $\pi_i : X \to X_i$ so that $\pi_i(x) = x_i$ where $x = \{x_i\}_{i\in I}$.

Definition. Suppose X and Y are topological spaces and $f : X \to Y$ is a function. Then f is said to be an open map if and only if $f(U)$ is open for each open subset U of X; and f is said to be a closed map if and only if $f(M)$ is closed for each closed subset M of X.

Exercise 5.7. Give examples of continuous onto functions that are not open or not closed.

Theorem 5.2. If $X = \Pi_{i \in I} X_i$ is a topological product space, then for each $i \in I$ the projection map $\pi_i(x)$ is continuous and open.

Theorem 5.3. Suppose for each positive integer i that X_i is a metric space. Then $\Pi_{i=1}^{\infty} X_i$ is a metric space.

Definition. The topological space X is said to be *metrizable* provided that there exists a distance function that generates the topology of X.

Theorem 5.4 A. Every completely separable normal space is metrizable.

Theorem 5.4 B. If X is a compact metric space then the metric for X is complete and bounded.

Definition. The space X is said to be *limit compact* if and only if every infinite subset of X has a limit point.

Theorem 5.5. If X is a metric space then X is compact if and only if it is limit compact.

Notes: the following two theorems are equivalent formulation of Zorn's lemma. Zorn's lemma is equivalent to the Axiom of Choice and the well-ordering property.

Theorem [Set inclusion version of Zorn's lemma.] Suppose that G is a collection of sets such that if J is a monotonic subcollection of G there there is a set in G which is a subset of every element of J. Then there is an element of G which contains no other element of G.

Definition. Suppose that A is a set; then the relation "<" is said to be a *partial order* if and only if:

1. $a \not< a$ for all $a \in A$, and
2. if $a < b$ and $b < c$ then $a < c$.

[In the literature a set with a partial ordered is called a partially ordered set or a po-set.]

Theorem [Partial order version of Zorn's lemma.] Suppose that A is a set with a partial order $<$ and B is a subset of A which is linearly ordered by $<$. Then there exists a maximal linearly ordered subset of A which contains B.