Topology Notes 07
Connectedness

Theorem 7.1. Suppose that H and K are disjoint closed point sets and G is a monotonic collection of compact point sets so that each element of G intersects both H and K but is not the union of two disjoint closed point sets one intersecting H and the other intersecting K. Then the common part $M = \cap G$ intersects both H and K but is not the union of two disjoint closed sets one intersecting H and the other intersecting K.

Definition. The point sets H and K are said to be *mutually separated* if and only if neither of them contains a point or a limit point of the other.

Definition. The point set M is said to be *connected* if and only if M is not the union of two mutually separated non-empty point sets.

Definition. A *continuum* is a compact and connected non-empty point set.

Exercise. If M is a non-degenerate connected set then every point of M is a limit point of M.

Theorem 7.2. If H and K are mutually separated point sets and M is a connected subset of $H \cup K$ then either $M \subset H$ or $M \subset K$.

Theorem 7.3. If M is connected then $\overline{M}$ is connected.

Theorem 7.4. Suppose that M is a connected set and G is a collection of connected sets so that M contains a point or a limit point of each element of G. Then $M \cup \cup_{g \in G} g$ is connected.

Theorem 7.5. If G is a monotonic collection of continua then $\cap G$ is a continuum.

Exercise. Is theorem 7.5 true if the word "continuum" is replaced with the word "connected"?

Definition. If H and K are two point sets then the continuum I is said to be *irreducible* from H to K if and only if I intersects H and K but no proper subcontinuum does so.

Theorem 7.6. If M is a compact set intersecting each of two disjoint closed point sets H and K but is not the union of two disjoint closed sets one intersecting H and the other intersecting K, then M contains a continuum irreducible from H to K.

Exercises.

a. If M is closed and not connected then M is the union of two disjoint closed non-empty sets.

b. If $f : X \to Y$ is continuous and $K \subset X$ is connected then $f(K)$ is connected.

c. If $f : X \to Y$ is continuous and $K \subset X$ is a continuum then $f(K)$ is a continuum.

d. Is there a connected set [or a continuum] which does not contain a homeomorphic copy of $[0, 1]$?

Theorem 7.7. Suppose that M is compact, H and K are closed subsets of M and no subcontinuum of M intersects both H and K. Then M is the union of two disjoint closed sets containing H and K respectively.

Theorem 7.8. If the continuum M is irreducible from the closed set H to the closed set K then $M - H \cup K$ is connected and every point of $H \cup K$ is a limit point of $M - H \cup K$.

Definition. By a *component* of a point set M is meant a maximal connected subset of M.

Theorem 7.9. Every point of a set belongs to a component of the set.

Theorem 7.10. Suppose that M is a continuum, O is an open set intersecting M but not containing M and C is a component of $M - O$. Then C intersects $Bd(O)$.

Theorem 7.11. If X is a nondegenerate connected normal Hausdorff space then X has at least c points. [A set with c points is equally numerous with the reals.]

Definition. The space X is said to be *locally connected at the point* p if and only if there is a local basis at p of connected sets. A set M is said to be *locally connected* if it is locally connected at each of its points.

Theorem 7.12. If X is locally connected and O is an open set in X then every component of O is open.

Exercise. Find an example of a continuum that is locally connected at none of its points.

Theorem 7.13. Suppose that X is a metric space, $M \subset X$ and M is the union of two mutually separated sets H and K. Then there exist disjoint open sets U and V containing H and K respectively.

Definition. Suppose that X is a space, $M \subset X$ and $f : X \to M$ is a continuous function. Then f is called a *retraction of* X *onto* M if and only if $f(x) = x$ for all $x \in M$. The subset M is called a *retract* of X.

Exercise. Characterize the retracts of the reals and of the plane.

Theorem 7.14. If X is a Hausdorff space then any retract of X is closed in X.

Theorem 7.15. If X is a connected Hausdorff space then any retract of X is connected.

Terminology. The term *map* will be used to mean a continuous function.

Theorem 7.16. If X is a locally connected Hausdorff space and f is a closed map from X onto the space Y, then Y is locally connected.

Exercise. Show that Theorem 7.16 is not true if the condition that f be closed is removed.

Question. What topological properties are preserved under retractions.

Question. Suppose that the sets H, $H \cap K$, and $H \cup K$ are all retracts of the space X. Then does it follow that K must be a retract of X.

Exercise. Show that if X is a normal Hausdorff space and I is a homeomorphic copy of the interval $[0, 1]$ lying in X, then I is a retract of X.