Topology 7510, Spring 2024
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Opening semester problems.

Problem 1. For this problem we will prove that no complete metric space is the union of countably many nowhere dense sets. So we suppose otherwise and that X is a complete metric space with complete metric d, and $M_1, M_2, M_3, \ldots$ is a sequence of nowhere dense sets whose union is X. Our strategy is to use the following facts:

$$\text{a.)} \qquad \bigcup_{i=1}^{N} M_i \quad \text{is nowhere dense.}$$

b.) We can select a sequence of points $x_1, x_2, x_3, \ldots$, which form a Cauchy sequence, and associated positive numbers $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ so that for each n, $\overline{B(x_n, \epsilon_n)} \cap \cup_{i=1}^{n} M_i = \emptyset$.

c.) Furthermore we can pick the points and positive numbers so that the limit ℓ of the sequence $x_1, x_2, x_3, \ldots$ is always inside of $\overline{B(x_n, \epsilon_n)}$.

d.) Argue that for each n, ℓ is not in M_n, which gives us our contradiction.

Hint: recall that if $\epsilon > 0$ then there exists an integer N so that

$$\sum_{i=N}^{\infty} \frac{1}{2^i} \quad < \quad \epsilon.$$

[In fact this is true for any convergent series.]

Problem 2. Prove that the real numbers is connected. [Hint: use the least upper bound axiom.]

Problem 3. Prove that if X is compact and $f : X \to \mathbb{R}$ is continuous then there us a point $t \in X$ so that $f(t)$ is a maximum [of the set $\{f(x)|x \in X\}$. [Hint: do it first for the case $X = [0, 1]$.]

Problem 4. An example of a Hausdorff space which is limit compact but not compact. Let M denote a well ordered set which is equally numerous with the reals. Let "$<$" denote that ordering, and give M the order topology.

We define an initial segment of M determined by the point m as the following set: $I(m) = \{x|x < m\}$. If there is an initial segment that is uncountable then let ω_1 be the least element of the set $\{m|I(m)$ is uncountable$\}$. If there is no initial segment that is uncountable, let $X = M$ otherwise let $x = I(\omega_1)$. Argue that:

a.) Every initial segment of X is countable, but that X is uncountable.

b.) X has infinitely many limit points.

c.) Definition: a set K is said to be cofinal in X if for each $x \in X$ there is a $k \in K$ so that $x < k$. Then the set of limit points of X is cofinal in X.

d.) No countable set is cofinal in X.

e.) If $K \subset X$ is infinite then K has a limit point.

f.) X is not compact.

[Extra credit: the set of limit points of X is order isomorphic to X.

Problem 5. If $M \subset \mathbb{R}$ is uncountable, then M contains one of its limit points.