Uncountable subsets of $\mathbb{R}$

First some lemmas on countability.

Lemma 1. If each of A and B is a countable set, then $A \cup B$ is countable.

Lemma 2. If each of A and B is a countable set, then $A \times B$ is countable.

Lemma 3. Suppose that for each $i \in \mathbb{N}$ that A_i is a countable set. Then $\cup_{i=1}^{\infty} A_i$ is countable.

Exercise. The set of rational numbers is countable.

Lemma 4. Suppose that for each $i \in \Gamma$, ϵ_i is a positive real number. Then, if Γ is uncountable, there exists an integer $n > 0$ so that the set $\{i \in \Gamma \mid \epsilon_i > \frac{1}{n}\}$ is uncountable.

Lemma 5. If $M \subset \mathbb{R}$ is uncountable, then M has a limit point.

Lemma 6. If $M \subset \mathbb{R}$ is uncountable, then M contains one of its limit points.