Well Ordering Theorem
Exercises 01
Math 7510

Definition. The relation $<$ is called an *order relation* means that it satisfies the following conditions:

If $x \neq y$ then either $x < y$ or $y < x$;
For all x, $x \not< x$;
If $x < y$ and $y < z$ then $x < z$.

Definition. Suppose that S is a set with an order relation $<$ and $M \subset S$. Then M has a least element means that there is an element $\ell \in M$ so that if $x \in M$ and $x \neq \ell$ then $\ell < x$.

Definition. Suppose that S is a set with an order relation $<$. Then S is said to be *well ordered* with respect to relation $<$ iff each subset of S has a least element.

Axiom of Choice: Suppose the G is a collection of sets. Then there is a function $F : G \to \cup G$ so that for each $g \in G$, $F(g) \in g$. The function F is called the choice function.

Well ordering theorem: The Axiom of Choice is equivalent to the statement that every set can be well ordered with respect to some ordering.

Exercise. Show the "easy" direction of the well ordering theorem.

For the following, assume the Axiom of Choice.

Definition. Suppose the S is an ordered set. Let B be the set to which b belongs if and only if:

there are two points x and y in S so that $b = \{t | x < t < y\}$,
there is a point x in S so that $b = \{t | x < t\}$, or
there is a point y in S so that $b = \{t | t < y\}$.

Then the *order topology* on S is the topology induced by letting B be a basis of open set.

To get a feeling of the property of well orderings, prove the following theorems.

Theorem. Suppose that the set S is well ordered with respect to the ordering $<$ and S has a last element. Then S with the order topology is compact.

Theorem. Suppose that the set S is well ordered with respect to the ordering $<$. Then there is no infinite sequence $\{x_i\}_{i=1}^{\infty}$ so that for each i we have $x_{i+1} < x_i$.

Definition. Suppose that the set S is well ordered with respect to the ordering $<$. Then the subset I of S is called an *initial segment* if and only if either there is a point $x \in S$ so that $I = \{t | t < x\}$ or $I = S$. The initial segment I is called a proper initial segment of S if $I \neq S$.

Definition. Suppose that each of K and M is an ordered set with orders $<_K$ and $<_M$ respectively. Then the function $f : K \to M$ is called an *order isomorphism* if and only if f is onto and for each pair of elements x and y in K so that $x <_K y$ we have $f(x) <_M f(y)$.

Theorem. Suppose that each of K and M is a well ordered set. Then either there is an order isomorphism from one of the sets onto the other or there is an order isomorphism from one of the sets onto an initial segment of the other.

Definition. The set X is said to be equally numerous with the set Y if and only if there is a one-to-one function of X onto Y.

Continuum Hypothesis [CH]: If X is an uncountable subset of the Reals $\mathbb{R}$ then X is equally numerous with $\mathbb{R}$.

Theorem [Assume CH and WOT]. There is a well ordering of the reals such that every proper initial segment is countable.

We wish to use the Axiom of Choice to prove the well ordering theorem.

Suppose then that X is a set, $\mathcal{G}$ is the collection of all non-empty subsets of X and $F : \mathcal{G} \to X$ is a choice function so that

$$F(g) \in g.$$

Thus F selects an element of g for each $g \in \mathcal{G}$. We wish to use F to create a well ordering of the points of X.

Exercise 1. yourself how you would use F to pick the element that will end up being the very first element of X. Then once you have selected the first element of X how would you select the second element of X.

Exercise 2. Suppose that M is an infinite well-ordered set. Argue that either M is order isomorphic to the positive integers or there is an initial segment of M that is order isomorphic to the positive integers.