Well ordering Lemmas

Lemma 1. Suppose that X is an uncountable well ordered set and there is an element of X that is preceded by uncountably many elements. Then X contains an uncountable well ordered initial segment so that every initial segment of it is countable

Lemma 2. Suppose that M is an uncountable well ordered set so that every initial segment is countable and X is an uncountable well ordered set. Then M is either order isomorphic to X or it is order isomorphic to an initial segment of X.

Construction. Consider the real numbers $\mathbb{R}$ with the usual order. Let S denote the set of all subsets of $\mathbb{R}$ so that $M \in S$ iff M is well ordered with respect to the usual order on the reals. Let $\sim$ be the relation on S so that $M_1 \sim M_2$ if and only if M_1 and M_2 are order isomorphic.

a. Show that $\sim$ is an equivalence relation.

Define $[M_1] < [M_2]$ if and only if there is an order isomorphism from M_1 onto an initial segment of M_2.

b. Show that $<$ is well defined.

c. Show that for each $[M]$ that there is an element $M' \in [M]$ so that $M' \subset [0, 1]$.

d. Show that $<$ is a well ordering on the equivalence classes of S.

e. Furthermore, show that the collection $\{[M] | M \in S\}$ is uncountable.