**Paracompact and Lindelöf spaces.**

Definition. Suppose that $X$ is a space, $M \subset X$ and $G$ is a collection of open sets which covers the set $M$. Then the collection $G'$ is said to be a *refinement* of $G$ covering $M$ if and only if for each point $x \in M$ there is an element of $G'$ that contains $x$ and lies in some element of $G$.

Definition. Suppose that $X$ is a space and $G$ is a collection of subsets of $X$. Then $G$ is said to be *locally finite* if and only if for each point $x \in X$ there is an open set $U$ containing $x$ so that $U$ intersects only finitely many elements of $G$.

Definition. The topological space $X$ is said to be *paracompact* if and only if it is Hausdorff and every collection of open sets that covers $X$ has a locally finite refinement of open sets that covers X.

Exercise. Show that if the word "refinement" is replaced with the word "subcover" then the definitions are not equivalent.

Exercise. Show that the space of real numbers $\mathbb{R}$ with the usual order topology is paracompact.

Exercise. Find an example of a space that is not paracompact.

Theorem 6.1. If $X$ is a compact space then it is paracompact.

Theorem 6.2. If $M$ is a closed subspace of the paracompact space $X$ then it is also paracompact.

Theorem 6.3. Every paracompact space is regular.

Theorem 6.4. Every paracompact space is normal.

Definition. The collection $G$ of subsets of the space $X$ is said to be *discrete* if and only if for each $x \in X$ there is an open set containing $x$ that intersects at most one element of $G$.

Definition. The space $X$ is said to be *collectionwise normal* if and only if whenever $\{M_i\}_{i \in I}$ is a discrete collection of closed sets then there exists a discrete collection of open sets $\{U_i\}_{i \in I}$ so that for each $i \in I$ we have $M_i \subset U_i$.

Exercise. Find an example of a normal space that is not collectionwise normal.

Theorem 6.5. Every paracompact space is collectionwise normal.

Definition. The topological space $X$ is said to be *Lindelof* if and only if for every collection $G$ of open sets that covers $X$ there is a countable subcollection $G' \subset G$ that covers $X$.

Exercise. Find an example of a space that is not Lindelof.

Theorem 6.6. Every completely separable space is Lindelof.

Theorem 6.7. Every separable paracompact space is Lindelof.