Upper semi-continuous collections and Open Maps

Definition. Suppose that X is a Hausdorff space and G is a collection of subsets of X. Then the collection G is said to be an *upper semi-continuous* collection mean that if $g \in G$ and U is an open set containing g, then there is an open set V containing g such that each member of G which intersects V lies in U.

Definition. If G is an upper semi-continuous collection of subsets of the topological space X and $X = \cup G$, then the decomposition space, denoted by X/G, is the space whose points are the elements of G and a basis for the topology of the space are those subsets of G each of whose union is open in X. Thus R is a basis element in X/G if and only if $\cup R$ is open in X.

For the theorems in this section suppose that G is an upper semi-continuous collection of sets filling up the Hausdorff space X. (I.e. $X = \cup G$.)

Theorem 8.1. No point of X belongs to two elements of G.

Theorem 8.2. No element of G contains a limit point of some other element of G.

Theorem 8.3. If $g \in G$ and D is an open set in X containing g, then the set of all members of G that lie in D is an open set in X/G.

Theorem 8.4. The element g of G is a limit point of the subset H of G if and only if g contains a limit point of $\cup H - g$.

Theorem 8.5. If K is a closed subset of X (with respect to the topology of X) and H is the set of all members of G which intersect K, then H is closed in X/G and $\cup H$ is closed in X.

Theorem 8.6. If H is a connected subset of G (with respect to the topology of X/G) and each member of H is connected (with respect to the topology of X) then $\cup H$ is connected.

Theorem 8.7. Suppose $H \subset G$. Then $\cup H$ is closed (in X) if and only if H is closed (in X/G).

Theorem 8.8. If $H \subset G$ and $\cup H$ is connected, then H is connected.

Theorem 8.9. If $H \subset G$ and $\cup H$ is compact, then H is compact; if H is compact and each member of H is compact, then $\cup H$ is compact.

Exercise. Suppose that G is an upper semi-continuous collection filling up the space X and X is T_n, a Moore space, or a metric space. Then is X/G a T_n space, a Moore space, or a metric space respectively?

Exercise. Repeat the above exercise in the case that each element of G is compact.

Exercise. Consider additional topological properties we have developed such as separability, first or second countability, paracompactness, etc.

Exercise. Suppose G is an upper semi-continuous collection of compact sets filling up the space X. Consider the function $f : X \to X/G$ defined by $f(x) = g$ iff and only if $x \in g$.

Then is f:

a. continuous?

b. an open map?

c. a closed map?

d. is compactness required for each of these?

Suppose $f : X \to Y$ is a continuous onto function and $G = \{f^{-1}(y)|y \in Y\}$. Then is G necessarily upper semi-continuous? What if we add one (or more of) the following conditions:

a. f is open.

b. f is closed.

c. $f^{-1}(y)$ is compact for each $y \in Y$.

Theorem 8.10. If X is metric then X/G is Hausdorff.

Theorem 8.11. Let $f : X \to X/G$ be defined by $f(p)$ is the element of G that contains p. Then f is continuous.

Theorem 8.12. If K is a compact subset of X, then the set of all members of G which intersect K is compact.

Theorem 8.13. If X is a metric space then X/G is normal.

Exercise. Is there:

a.) an upper semi-continuous collection of non-degenerate compact sets that fills up the plane?

b.) an upper semi-continuous collection of non-degenerate compact sets that fills up the unit interval?

c.)an upper semi-continuous collection of non-degenerate continua that fills up the plane?

d.) an upper semi-continuous collection of non-degenerate continua that fills up the unit interval?