Homotopy, the Fundamental Group

Definition. Let X be a topological space and let I denote the unit interval $[0, 1]$ with the usual topology. Then a continuous map $f : I \to X$ is called a *path* in X. The point $p = f(0)$ is called the initial or first point of the path, the point $q = f(1)$ is called the final or last point of the path and f is said to be a path from p to q.

Definition. The space X is said to be pathwise connected if and only if for each pair of points $\{x, y\}$ of X there is a path from x to y.

Definition. Let X and Y be topological spaces and let $f, g : X \to Y$ be continuous functions. Then the f is said to be *homotopic* to g if and only if there is a continuous map $F : X \times I \to Y$ so that:

$F(x, 0) = f(x)$ for all $x \in X$ and

$F(x, 1) = g(x)$ for all $x \in X$.

The map F is called a *homotopy* and the notation "$\simeq$" is used to indicate when functions are homotopic; thus $f \simeq g$ means that f is homotopic to g. A map the is homotopic to a constant map is said to be *nullhomotopic*.

Exercise 9.1. Show that $\simeq$ is an equivalence relation.

Notation. If X and Y are topological spaces then $[X, Y]$ denotes the collection of homotopy equivalence classes.

Notation. If f is a path, then $[f]$ denotes the set of paths homotopic to f. Thus $[I, X] = \{[f]|$ f is a path in $X\}$.

Definition. Suppose that f is a path from x to y in X and g is a path from y to z in X. Then $f * g$ denotes the function h defined by:

$h(t) = f(2t)$ for $t \in [0, \frac{1}{2}]$,

$h(t) = g(2t - 1)$ for $t \in (\frac{1}{2}, 1]$.

Observe that h is continuous at $t = \frac{1}{2}$ so calling it a "path" is valid. If f is a path $[f]$ denotes the equivalence class of f: $[f] = \{g|g \simeq f\}$.

Notation. Let us assume that when we refer to $f * g$ that f and g are paths and that the last point of f is the first point of g.

Exercise 9.2. [Using the notation in the above definition]: Show that $f * g$ is a path from x to z .

Exercise 9.3. Define $*$ on the collection of equivalence classes $[I, Y]$ of paths by $[f] * [g] = [f * g]$. On this set of equivalence classes, determine:

a. If the operation $*$ is well defined.
b. If the operation $*$ is associative.
c. If the operation $*$ has left and right identities.
d. If the operation $*$ has corresponding inverses.

Definition. The space X is said to be *contractible* if and only if the identity function on X is nullhomotopic.

Theorem 9.1. If X is contractible then it is pathwise connected.

Definition. Suppose that f and g are two paths $f, g[0,1] \to X$ then f is said to be *path-homotopic* to g if $f(0) = g(0)$, $f(1) = g(1)$ and there exists a homotopy $H : I \times I \to X$ so that:

$H(t, 0) = f(t)$, $H(t, 1) = g(t)$,
$H(0, t) = f(0)$ for all $t \in I$,
$H(1, t) = f(1)$ for all $t \in I$.

Thus a path-homotopy keeps the endpoints of the paths fixed. Let us use the notation $f \simeq_P g$ to denote the fact that f is path-homotopic to g.

Exercise: Show that $\simeq_P$ is an equivalence relation.

Definition. A *loop* is a path in which the initial point is also the final point; this point is called the base point of the loop.

Theorem 9.2. Suppose that X is a topological space and $a \in X$. Let $\pi_1(X, a)$ denote the collection of path-homotopy equivalence classes of all loops based at a. Then $\pi_1(X, a)$ with operation $*$ is a group.

Definition. The group $\pi_1(X, a)$ is called the fundamental group of X based at the point a.

Exercise 9.4. Calculate $\pi_1(X, a)$ for:
a. $X = [0, 1]$,
b. $X = \mathbb{R}^2$,
c. X is the unit circle $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$.

Theorem 9.3. Let $a, b \in X$ and suppose that there is a path from a to b. Then the groups $\pi_1(X, a)$ and $\pi_1(X, b)$ are isomorphic.

Theorem 9.4. If X is contractible then $\pi_1(X, a)$ is the trivial group.

Theorem 9.5. Suppose X and Y are topological spaces, $a \in X$ and $f : X \to Y$ is a continuous map. Define $f_* : \pi_1(X, a) \to \pi_1(Y, f(a))$ by $f_*([h]) = [f \circ h]$. Then f_* is a homomorphism.

Theorem 9.6. Suppose X and Y are homeomorphic pathwise connected topological spaces, $x \in X$ and $y \in Y$. Then $\pi_1(X, x)$ and $\pi_1(Y, y)$ are isomorphic.

Exercise 9.5. Let X be a topological space.
a. Show that following holds when the expressions are meaningful:

$$(f \circ g)_* = f_* \circ g_*.$$

b. Show that if i is the identity map on X then i_* is the identity homomorphism.

Theorem 9.7. The unit interval is contractible.

Exercise. 9.6. Is the real line contractible?

Definition. The subset M of Euclidean n-space is said to be convex if and only if for each pair of points in M the straight line segment containing the points is also in M. (In vector notation if $A \in M$ and $B \in M$ then $\{tA + (1 - t)B \mid 0 \leq t \leq 1\} \subset M$.)

Theorem 9.8. A compact convex subset of $\mathbb{R}^n$ is contractible.

Question. Is compactness necessary in theorem 9.8?

Exercise 9.7. Let X be the punctured plane, $X = \mathbb{R}^2 - \{(0,0)\}$; so that X is the plane with the origin removed. Let I denote the unit interval. Let S denote the unit circle $S = \{(\cos(\theta), \sin(\theta)) \mid 0 \leq \theta \leq 2\pi\}$; for each θ let P_θ denote the point $(\cos(\theta), \sin(\theta)) \in S$. Determine which of the following function are nullhomotopic.

a. $f : I \to X$ such that $f(t) = (t, 1 + t^2)$.
b. $f : I \to X$ such that $f(t) = (\cos(\frac{t}{2\pi}), \sin(\frac{t}{2\pi}))$.
c. $f : S \to X$ such that $f(P_\theta) = (\cos(\theta), \sin(\theta))$.
d. $f : S \to X$ such that $f(P_\theta) = (\cos(\theta) + 5, \sin(\theta) + 5)$.