Covering Spaces.

Notation: we will frequently use the word "map" to mean continuous function.

Definition. Suppose that Y is a space. Then the space X is said to be a *covering space* for Y if and only if there is a map $p : X \to Y$ so that for each $y \in Y$ there is an open set U containing y so that $p^{-1}(U)$ is the union of disjoint open sets $\{W_\alpha\}_{\alpha\in\Gamma}$ and for each $\alpha \in \Gamma$ the map $p|_{W_\alpha}$, this is the map p restricted to the open set W_α, is a homeomorphism of W_α onto U. The map p is called a *covering map*.

Exercise 10.1.

a. Let $Y = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ be the unit circle and let $p : \mathbb{R} \to Y$ be defined by $p(\theta) = (\cos(\theta), \sin(\theta))$. Then $\mathbb{R}$ is a covering space for Y and p is a covering map.

b. Let $\mathbb{C}$ denote the complex plane, let $X = \{z \in \mathbb{C} | \ \|z\| = 1\}$ and let $p : X \to X$ be defined by $p(z) = z^2$. Then X is a covering space for itself and p is a covering map. [Hint: recall that if $z = \cos(\theta) + i\sin(\theta)$ then $z^2 = \cos(2\theta) + i\sin(2\theta)$.]

Exercise 10.2.

a. Let S^1 denote the circle. Then the surface of the torus T is the space $S^1 \times S^1$. Show that $\mathbb{R}^2$ is a covering space for T.

b. Show that the Annulus is a covering space for the Moebius strip.

c. Is $\mathbb{R}^2$ is a covering space for the surface of the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$?

Definition. If X is a covering space for Y with covering map p and $h : S \to Y$ is a map of the space S into Y then the map $\widetilde{h} : S \to X$ is called a *lifting* of h to X if and only if $p(\widetilde{h}(x)) = h(x)$ for all $x \in S$.

Theorem 10.1. Let X be a covering space for Y with covering map p and let $p(x_0) = y_0$; let $h : I \to Y$ be a path with initial point y_0. Then there is a unique lifting $\widetilde{h}$ of h to X so that $\widetilde{h}(0) = x_0$.

Theorem 10.2. Suppose that X is a covering space for Y with covering map p and with $p(x_0) = y_0$; suppose that $F : I \times I \to Y$ is a path-homotopy so that $F(0,0) = y_0$. Then there is a unique lifting $\widetilde{F}$ of F so that $\widetilde{F}(0,0) = x_0$.

Theorem 10.3. Suppose that X is a covering space for Y with covering map p and with $p(x_0) = y_0$. Suppose that f and g are path-homotopic paths in Y from y_0 to the point y_1 and that each of $\widetilde{f}$ and $\widetilde{g}$ are their unique liftings to paths in X respectively with initial point x_0. Then $\widetilde{f}$ and $\widetilde{g}$ are path-homotopic in X and $\widetilde{f}(1) = \widetilde{g}(1)$.

Theorem 10.4. Let S^1 denote the unit circle and let $a \in S^1$. Then $\pi_1(X, a)$ is isomorphic to the the integers $\mathbb{Z}$ with the group operation of standard addition.

Theorem 10.5. Suppose that X is a covering space for Y with covering map p and with $p(x_0) = y_0$. For each $[f] \in \pi_1(Y, y_0)$ let $\widetilde{f}$ denote the unique lift of f with $f(0) = x_0$. Then ϕ defined by $\phi([f]) = \widetilde{f}(1)$ is a well defined function from $\pi_1(Y, y_0)$ to $p^{-1}(y_0)$.

Theorem 10.6. Suppose that X is a path-connected covering space for Y with covering map p with $p(x_0) = y_0$ and ϕ is as in Theorem 10.5; let $G = \pi_1(Y, y_0)$ and let $H = p_*(\pi_1(X, x_0))$. Then $\Phi : G/H \to p^{-1}(y_0)$ defined by $\Phi(H[f]) = \phi([f])$ is well defined, one-to-one and onto.

Definition. Let X be a topological space and let $M \subset X$. Then M is called a *retract* of X if and only if there is a continuous function $f : X \to M$ so that $f(x) = x$ for all $x \in M$. The function f is called a *retraction.*

Theorem 10.7. Suppose that M is a retract of X and j is the "inclusion" map $j : M \to X$ (where $j(x) = x \; \forall x \in M$.) Let $x_0 \in M$ then the homomorphism $j_* : \pi_1(M, x_0) \to \pi_1(X, x_0)$ is one-to-one.

Theorem 10.8. [van Kampen] Suppose that the topological space X is the union of two open sets U and V so that $U \cap V$ is pathwise connected and $a \in U \cap V$. Let j^U and j^V denote the inclusion maps of U and V into X respectively. Then the subgroups $j^U_*(\pi_1(U, a))$ and $j^V_*(\pi_1(V, a))$ generate the group $\pi_1(X, a)$.

Theorem 10.9. Suppose that X and Y are pathwise connected topological spaces and $x \in X, y \in Y$. Then $\pi_1(X \times Y, \{(x,y)\}) \simeq \pi_1(X,x) \times \pi_1(Y,y)$. [Hint look at the homomorphisms introduced by the projection maps onto each coordinate.]