Inverse Limit Spaces

Definition: For each positive integer i let X_i be a topological space and let $f_i : X_{i+1} \to X_i$ be a continuous map. Then the inverse limit space $\mathbf{X} = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ is defined as follows.

The element $P = \{P_i\}_{i=1}^{\infty}$ is a point of $\mathbf{X}$ provided for each positive integer i, $P_i \in X_i$ and $f_i(P_{i+1}) = P_i$. If U_i is a subset of X_i then $\overleftarrow{U_i} = \{P \in \mathbf{X} | P_i \in U_i\}$. The collection $\mathcal{B} = \{\overleftarrow{U_i} | U_i$ is open in $X_i; i = 1, 2, ...\}$ is a basis for the topology for X.

Questions. Suppose that $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ is an inverse limit space. Suppose further that for each positive integer i the space X_i has property $\mathcal{P}$ then does it follow that X also has property $\mathcal{P}$ where property $\mathcal{P}$ is one of the following.

0. Non-empty.
1. Hausdorff.
2. Compact.
3. Connected.
4. Metric.
5. A continuum.
6. Separable.
7. Completely separable or second countable.
8. Normal.

Questions 2: Repeat the above questions with onto bonding maps.

Theorem 11.1. Suppose that $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ is an inverse limit space so that there is an integer N and for each $n \geq N$ the function f_n is an onto homeomorphism. Then X is homeomorphic to X_N.

Definition. Suppose that $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ is an inverse limit space. If $i < j$ then f_i^j denotes the functions $f_i \circ f_{i+1} \circ ... \circ f_{j-1} : X_j \to X_i$. Note$f_i^{i+1} = f_i$.

Theorem 11.2. Suppose that $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ is an inverse limit space and that $\{n_i\}_{i=1}^{\infty}$ is an increasing sequence of positive numbers, $g_i = f_{n_i}^{n_{i+1}}$ and $Y = \varprojlim\{X_{n_i}, g_i\}_{i=1}^{\infty}$. Then X is homeomorphic to Y.

Theorem 11.3. The inverse limit space $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ is a subspace of the product space $\Pi_{i=1}^{\infty} X_i$.

Question. Under the hypothesis of the previous theorem, under what condition is the inverse limit space a closed subspace of the product space. An answer to this question may provide answers to the questions concerning Property $\mathcal{P}$ above.

Theorem 11.4. For each positive integer n let $X_n = \{1, 2, ..., 2^n\}$ with the discrete topology and let $f_n : X_{n+1} \to X_n$ be the function defined by:

$f_n(i) = i$ for $1 \leq i \leq 2^n$;

$f_n(i) = 2^{n+1} - i + 1$ for $2^n < i \leq 2^{n+1}$.

Then $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ is homeomorphic to the Cantor set.

Theorem 11.4'. For each integer n let X_n denote the two point topological space $\{0, 1\}$. Then the product space $X = \Pi_{i=1}^{\infty} X_i$ is homeomorphic to the Cantor set.

Theorem 11.5. Suppose that $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ is an inverse limit space and for each integer n the point $P^n = \{P_i^n\}_{i=1}^{\infty}$ is an element of X. Then the point $Q = \{Q_n\}_{n=1}^{\infty}$ in X is a sequential limit point of the set $\{P^j\}_{j=1}^{\infty}$ if and only if for each integer k Q_k is a sequential limit point of the set $\{P_k^n\}_{n=1}^{\infty}$ in the space X_k.

Exercise 11.1. Suppose that $X = \Pi_{i=1}^{\infty} X_i$ is a topological product space, $Y_n = \Pi_{i=1}^{n} X_i$ and $f_n : Y_{n+1} \to Y_n$ is defined by $f_n(x_1, x_2, ..., x_{n+1}) = (x_1, x_2, ..., x_n)$. Then X is homeomorphic to $\varprojlim\{Y_n, f_n\}_{n=1}^{\infty}$.

Definition. Suppose that X is a continuum. Then the point p is a cut-point of X means that $X - \{p\}$ is not connected.

Definition. Let $p \in X$ then the set Cp defined as the set to which q belongs if and only if there is a path in X containing both p and q is called the path component of X at the point p.

Theorem. If p and q are points in X and $Cp \cap Cq \neq \emptyset$ then $Cp = Cq$.

Definition. Let $p \in X$ then the set Ap defined as the set to which q belongs if and only if there is an arc lying in X containing both p and q is called the arc component of X at the point p.

Theorem. If p and q are points in X and $Ap \cap Aq \neq \emptyset$ then $Ap = Aq$.

Exercise 11.2. Let I denote the unit interval $[0, 1]$ and let f be the function defined as follows:

Let $f(t) = \left\{ \begin{array}{c} \frac{3}{2}t, 0 \leq t \leq \frac{2}{3} \\ \frac{5}{3} - t, \frac{2}{3} \leq t \leq 1 \end{array} \right\}$.

For each positive integer i let $X_i = I$ and $f_i = f$ and let:

$$X = \varprojlim \{X_i, f_i\}_{i=1}^{\infty} = \varprojlim \{I, f\}_{i=1}^{\infty}.$$

Then show that the inverse limit space has the following properties:

1. X has two arc-components.
2. One arc-component is dense in X.
3. One arc-component is a topological interval.
4. X is irreducible between some two points.
5. The set of cut points is dense in X.
6. X is not locally connected at some point.
7. X is not locally connected at each point of one of the arc - components.

Hints. The function f has two fixed points; can anything special be said about that number? The function f has some "period 2" points; does that give us any information about the inverse limit?

Theorem 11.6. Suppose that $X = \varprojlim \{X_i, f_i\}_{i=1}^{\infty}$ is an inverse limit space and for each integer n the space X_n is compact. Then X is non-empty and compact.

Theorem 11.7. Suppose that $X = \varprojlim \{X_i, f_i\}_{i=1}^{\infty}$ is an inverse limit space and for each integer n the space X_n is a continuum. Then X is a continuum.