1 Fractal processes

Recall that a (probability distribution of) Gaussian process is fully described by its mean function $M(t) = \mathsf{E}\, X(t)$ and its covariance function which, for zero-mean processes, is given by $C(s,t) = \mathsf{E}\, X(s)X(t)$. The existence of a process with a given $C(s,t)$ is ensured either by Bochner's Theorem, or by a specific construction. Also, within the class of processes with stationary increments, the variance function $V(t) = \mathsf{E}\, |X(t)|^2$ determines the covariance:

$$C(s,t) = \frac{V(t) + V(s) - V(t-s)}{2}.$$

1.1 Fractional Brownian Motion - FBM

For Brownian Motion, $V(t) = |t|$. Immediately, by comparing the covariance functions,

$$B(\lambda t) \stackrel{\mathcal{D}}{=} \lambda^{1/2} B(t), \qquad \text{for every } \lambda > 0,$$

so the time-scaling of Brownian motion is equivalent to the range-scaling, with the exponent 1/2, regarding the finite dimensional distributions. In general, we say that a process $X(t)$ is **self-similar with exponent** H if

$$X(\lambda t) \stackrel{\mathcal{D}}{=} \lambda^{H} X(t), \qquad \text{for every } \lambda > 0. \tag{1}$$

Its variance function would be then, up to a constant,

$$V_H(t) = c_H t^{2H} \tag{2}$$

A self-similar zero-mean Gaussian process with independent increments and the above variance function is called a **fractional Brownian motion**. Does such process exist? If we knew that the function

$$C(s,t) = \frac{|t|^{2H} + |s|^{2H} - |t-s|^{2H}}{2}. \tag{3}$$

was semi-positive definitive, Bochner's theorem would give the affirmative answer.

Yet, a suitable construction seems to be the safest approach. Recall the second construction of the Brownian integral $B(f) = \int f\, dB$. It can be mimicked under much weaker assumptions. Below, T is an interval of the real line (a closed inerval, the half-line $[0,\infty)$, etc.).

Theorem 1 *Let $Z(t)$ be a real (or complex) stochastic process on T with mean zero, finite variance, and orthogonal (i.e., uncorrelated) increments. That is,*

$$\mathsf{E}\, Z(t) = 0, \qquad F(t) = \mathsf{E}\, |Z(t)|^2 < \infty, \qquad \mathsf{E}\Big(Z(b) - Z(a)\Big)\overline{\Big(Z(d) - Z(c)\Big)} = 0 \quad \textit{when } a < b < c < d\ .$$

Then, $F(t)$ is a non-decreasing function, and, for every square-integrable real (or complex) function f, i.e.,

$$\int_T |f(t)|^2\, dF < \infty,$$

the random variable

$$Z(f) = \int_T f(t)\, Z(dt)$$

is well defined. Further, $Z(f)$ is a linear isometry.

Proof. No differentiation is involved in the construction. In the first step we consider the step function and the natural assignment

$$f = \sum_k c_k \, \mathbb{1}_{(t_{k-1}, t_k]} \quad \mapsto \quad Z(f) = \sum_k c_k \Big(Z(t_k) - Z(t_{k-1}) \Big).$$

The orthogonality of increments ensure that

$$\mathsf{E}\,|Z(f)|^2 = \sum_k |c_k|^2 \Big(F(t_k) - F(t_{k-1}) \Big) = \int_T |f|^2 dF,$$

with the Stieltjes-Lebesgie integral at the end. Note that this integral is well defined, because $F(t)$ is a non-decreasing function. Indeed, for $s < t$,

$$F(t) = \mathsf{E}\,|Z(s) + Z(t) - Z(s)|^2 = \mathsf{E}\,|Z(s)|^2 + \mathsf{E}\,|Z(t) - Z(s)|^2 \geq \mathsf{E}\,|Z(s)|^2 = F(s).$$

The obtained formula constitutes an isometry between a dense subspace of $L^2(T, F)$ and a subspace of $L^2(\Omega, \mathsf{P})$. The isometry can be extended over the entire $L^2(T, F)$. ■

We have already obtained a large family of processes with independent increments, e.g., the Brownian motion $B(t)$, the centered Poisson process $N(t) - \mathsf{E}\,N(t) = N(t) - \lambda t$, and the symetrized Poisson process (with jumps ± 1 with probability $1/2$ in lieu of unit positive jumps). The replacement of the 'clock' $t \mapsto L(t)$, where $L(t)$ is a monotonic function, or the replacement of Rademacher functions by any independent symmetric random variables, will preserve the orthogonal scatter.

Example. Let $Z(t)$ (real or complex) satisfy the hypotheses of the theorem. The complex process

$$S(t) = \int_T e^{iut} Z(du) \tag{4}$$

exists if, and only if,

$$range(F) = \sup_T F(t) - \inf_T F(t) = \int_T F(du) < \infty.$$

Let's compute its covariance function. Symbolically

$$\mathsf{E}\,Z(du)\overline{Z(du)} = \mathsf{E}\,|Z(du)|^2 = dF(u), \quad \mathsf{E}\,Z(du)\overline{Z(dv)} = 0, \quad \text{when } u \neq v.$$

Whence

$$\mathsf{Cov}\,(S(s), S(t)) = \mathsf{E}\,S(s)\overline{S(t)} = \mathsf{E}\Big(\int_T e^{ius}\, Z(du) \cdot \int_T e^{-ivt}\, \overline{Z}(dv) \Big) = \int_T e^{iu(s-t)}\, dF(u).$$

In other words, the process (4) is **stationary** [1]. That is, any translation of the parameter $t \mapsto t + a$ does not affect the covariance.

Example Let's formally integrate the process $S(t)$ given by (4), and suppose that we can switch the order of integration. We obtain then

$$W(t) = \int_0^t S(y)\, dy = \int_T \frac{e^{iut} - 1}{iu} Z(du) \tag{5}$$

[1] in the wide sense, in contrast to the stationarity in the narrow sense, involving the finite dimensional distributions

Even if one questions the validity of the above manipulations, the above process is well defined by Theorem 1 if, and only if,

$$\int_T \frac{\left|e^{iut}-1\right|^2}{|u|^2}\, dF(u) < \infty$$

The modulus is easily computable, and equals $2(1-\cos ut)$. So the process $W(t)$ makes sense as long as

$$\int_T \frac{1-\cos ut}{|u|^2}\, dF(u) < \infty, \quad \text{for every } t. \tag{6}$$

Since $1-\cos x \leq 1 \wedge x^2/2$, then to obtain $W(t)$ it is enough to assume that

$$\int_t \left(1 \wedge u^2 t^2\right) dF(u) < \infty.$$

The latter integral can be split into two parts, one for small u, the other - for large u. So, a sufficient condition consists of two requirements:

$$\int_{|u|\leq \epsilon} u^2\, dF(u) < \infty \quad \text{and} \quad \int_{|u|\geq K} dF(u) < \infty,$$

for every ϵ, K (we replace $\epsilon = 1/t$, when $t \to \infty$ or $K = 1/t$, when $t \to 0$).

Corollary 2 *Any process $Z(t) = Z_1(t) + iZ_2(t)$ on $\mathbb{R}$ satisfying the hypotheses of Theorem 1, whose related function $F(t)$ satisfies* (6)*, defines a complex process $W(t) = X(t) + iY(t)$ with stationary increments by formula* (5)*. The real and imaginary parts are then given by the formulas*

$$\begin{aligned} X(t) &= \int_{\mathbb{R}} \left(\cos ut - 1\right) Z_1(du) - \int_{\mathbb{R}} \sin ut\, Z_2(du) \\ Y(t) &= \int_{\mathbb{R}} \sin ut\, Z_1(du) + \int_{\mathbb{R}} \left(\cos ut - 1\right) Z_2(du) \end{aligned}$$

If Z_1 and Z_2 have the same distribution and increments of Z_1 and Z_2 over disjoint intervals are independent, then X and Y are equi-distributed real processes with stationary increments, with the variance function

$$v(t) = 2\int_{\mathbb{R}} \left(1-\cos ut\right) dF(u).$$

Pick now a standard complex Brownian motion $B(t) = B_1(t) + iB_2(t)$, with independent real and imaginary parts. Put

$$W(t) = \int_T \frac{e^{iut}-1}{|u|^{H+1/2}}\, B(du) \tag{7}$$

Clearly, this complex process is mean zero Gaussian whose real and imaginary parts have the same distribution, are Gaussian, and have stationary increments.

$$\begin{aligned} X(t) &= \int_{\mathbb{R}} \frac{\cos ut - 1}{|u|^{H+1/2}}\, B_1(du) - \int_{\mathbb{R}} \frac{\sin ut}{|u|^{H+1/2}}\, B_2(du) \\ Y(t) &= \int_{\mathbb{R}} \frac{\sin ut}{|u|^{H+1/2}}\, B_1(du) + \int_{\mathbb{R}} \frac{\cos ut - 1}{|u|^{H+1/2}}\, B_2(du) \end{aligned}$$

Its variance function is

$$2\int_{\mathbb{R}} \frac{\cos ut - 1}{|u|^{2H+1}}\, du = |t|^{2H} \cdot 2\int_{\mathbb{R}} \frac{\cos v - 1}{|v|^{2H+1}}\, dv.$$

The integral is finite iff

$$0 < H \leq 1.$$

Further, the processes $W(t)$, $X(t)$, $Y(t)$ are self-similar with the exponent H.