Some bounds for the spectral radius of the Hadamard product of matrices

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in honor of Prof. Jean H. Bevis
Some bounds for the spectral radius of the Hadamard product of two nonnegative matrices are given. Some results involve $M$-matrices.

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I. Introduction

Given $A, B \in \mathbb{C}_{n \times n}$, the Hadamard product of $A$ and $B$ is

$$A \odot B = (a_{ij}b_{ij}).$$

Example 1.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, A \odot B = \begin{pmatrix} 2 & -2 \\ 3 & 0 \end{pmatrix}$$

If $A, B \geq 0$, then

$$\rho(A \odot B) \leq \rho(A)\rho(B),$$

where $\rho(A)$ is the spectral radius of $A$.

Proof:

- The Kronecker product $A \otimes B \geq 0$ since $A, B \geq 0$.

- $\rho(A \otimes B) = \rho(A)\rho(B)$. 
• $A \odot B$ is a principal submatrix of $A \otimes B$.

• Apply monotonicity of the Perron root.

By monotonicity of the Perron root of the $B \geq 0$,
\[
\max_{i=1,\ldots,n} b_{ii} \leq \rho(B). \tag{2}
\]
The lower bound is attained when $B \geq 0$ is diagonal.

Is it possible to have a better bound like the following for $\rho(A \odot B)$, where $A, B \geq 0$?
\[
\rho(A \odot B) \leq \rho(A) \max_{i=1,\ldots,n} b_{ii} \tag{3}
\]
Example 2.

\[ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad A \circ B = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \]

Evidently

\[ \rho(A) = 1, \quad \rho(B) = 3, \quad \rho(A \circ B) = 2, \]

and

\[ \max_{i=1,2} b_{ii} = 1. \]

So

\[ \rho(A \circ B) = 2 \nless 1 = \rho(A) = \max_{1 \leq i \leq 2} b_{ii}. \]
Goal: provide a necessary condition for (3) to be valid.

It turns out the condition is satisfied by an important class of matrices called inverse $M$-matrices.
II. Some bounds and diagonal dominance

A matrix $A$ is said to be \textit{diagonally dominant of its row entries} (respectively, of its columns entries) if

$$|a_{ii}| \geq |a_{ij}| \quad \text{(respectively } |a_{ii}| \geq |a_{ji}|)$$

for each $i = 1, \ldots, n$ and all $j \neq i$.

\textbf{Example 3.}

$$A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

is diagonally dominant of its column entries but not of its row entries.

Similarly we define \textit{diagonally subdominant} of its row entries (respectively, of its columns entries) by reversing the inequalities. Strict diagonal dominance is defined similarly.
Theorem 1. Let $A \geq 0$, $B \geq 0$ be $n \times n$ non-negative matrices. If there exists a positive diagonal $D$ such that

1. $DBD^{-1}$ is diagonally dominant of its column (or row) entries, then
   \[ \rho(B) \leq \text{tr} B, \]  
   (4)
   and
   \[ \rho(A \circ B) \leq \rho(A) \max_{i=1,...,n} b_{ii}, \]  
   (5)

2. $DBD^{-1}$ is diagonally subdominant of its column (or row) entries, then
   \[ \rho(B) \geq \text{tr} B \]
   and
   \[ \rho(A) \min_{i=1,...,n} b_{ii} \leq \rho(A \circ B). \]
Proof:

• \( A \circ (DBD^{-1}) = D(A \circ B)D^{-1} \) and hence \( \rho(A \circ B) = \rho(A \circ (DBD^{-1})) \).

• \( \text{diag } B = \text{diag } DBD^{-1} \). So we may assume that \( B \) is diagonally dominant of its column (or row) entries.

• \( A \circ B \leq A \text{ diag } (b_{11}, \ldots, b_{nn}) \leq A \max_{i=1,\ldots,n} b_{ii} \).

• By the monotonicity of the Perron root

\[
\begin{align*}
\rho(A \circ B) & \leq \rho(A \text{ diag } (b_{11}, \ldots, b_{nn})) \\
& \leq \rho(A) \max_{i=1,\ldots,n} b_{ii},
\end{align*}
\]

which yields (5) immediately.

• To obtain (4), set \( A = J_n \) in the first inequality of (6). Then

\[
\rho(B) \leq \rho(J_n \text{ diag } (b_{11}, \ldots, b_{nn})) = \text{tr } B,
\]
since \( \text{rank} \left( J_n \text{ diag} (b_{11}, \ldots, b_{nn}) \right) \leq 1 \).

**Remark 1.** 1. The upper bound \( \text{tr} B \) in (4) is attained by \( B = J_n \).

2. Though \( \max_{i=1,\ldots,n} b_{ii} \leq \rho(B) \) is true for \( B \geq 0, \rho(B) \leq \text{tr} B \) in (4) may not hold if the assumption in the theorem is dropped, for example, the irreducible

\[
B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

3. It is not true that if \( A \geq 0 \) and \( B \geq 0 \) are both diagonally dominant of its (column) row entries, then

\[
\rho(A \circ B) \leq \max_{i=1,\ldots,n} a_{ii} \max_{i=1,\ldots,n} b_{ii}.
\]

For example,

\[
A = \begin{pmatrix} 2 & 1 \\ 1 & 1.5 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, A \circ B = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix},
\]
with

\[ \rho(A \circ B) \approx 4.6180 > 4 = \max_{i=1,2} a_{ii} \max_{i=1,2} b_{ii}. \]

**Corollary 1.** Let \( B \geq 0 \) be an \( n \times n \) nonnegative matrix. If there exists a positive diagonal matrix \( D \) such that \( DBD^{-1} \) is diagonally dominant of its column (or row) entries, then

\[ \max\{\rho(A \circ B) : A \geq 0, \rho(A) = 1\} = \max_{i=1,...,n} b_{ii}. \]

**M-matrices:**

\( Z_n := \{ A \in \mathbb{R}^{n \times n} : a_{ij} \leq 0, i \neq j \} \).

A matrix \( A \in Z_n \) is called an **\( M \)-matrix** if there exists an \( P \geq 0 \) and \( s > 0 \) such that

\[ A = sI_n - P \quad \text{and} \quad s > \rho(P), \]
\( \mathcal{M}_n = \) the set of all \( n \times n \) nonsingular \( M \)-matrices.

The matrices in \( \mathcal{M}_n^{-1} := \{A^{-1} : A \in \mathcal{M}_n \} \) are called inverse \( M \)-matrices.

Known: Given \( A \in \mathbb{Z}_n \).

\( A \in \mathcal{M}_n^{-1} \iff A \) is nonsingular and \( A \geq 0 \).

**Corollary 2.** Let \( A \geq 0, B \geq 0 \) be \( n \times n \) non-negative matrices. If \( B \in \mathcal{M}_n^{-1} \), then

\[
\rho(B) \leq tr B,
\]

and

\[
\rho(A \circ B) \leq \rho(A) \max_{i=1, \ldots, n} b_{ii}.
\]

Hence if \( B \in \mathcal{M}_n^{-1} \), then

\[
\max\{\rho(A \circ B) : A \geq 0, \rho(A) = 1\}
= \max_{i=1, \ldots, n} b_{ii}.
\]
Proof:

- Since $B^{-1} \in \mathcal{M}_n$, there exists a positive diagonal $D$ such that $DB^{-1}D^{-1}$ is strictly row diagonally dominant.

- The inverse $DBD^{-1}$ of $DB^{-1}D^{-1}$ is strictly diagonally dominant of its column entries.

- Then apply Theorem [1] (1).
III. A sharper upper bound when $B^{-1}$ is an $M$-matrix

The inequality in Corollary 2 has a resemblance of a known result which asserts that if $A, B \in \mathcal{M}_n$, then

$$\tau(A \circ B^{-1}) \geq \tau(A) \min_{i=1,...,n} \beta_{ii}, \quad (7)$$

where

$$\tau(A) = \min\{\text{Re} \lambda : \lambda \in \sigma(A)\},$$

and $\sigma(A)$ is the spectrum of $A \in \mathcal{M}_n$.

**Known:**

$$\tau(A) = \frac{1}{\rho(A^{-1})}$$

and is a **positive** eigenvalue of $A \in \mathcal{M}_n$. 
The number $\tau(A)$ is often called the minimum eigenvalue of the $M$-matrix $A$. Indeed

$$\tau(A) = s - \rho(P),$$

if $A = sI_n - P$ where $s > \rho(P)$, $P \geq 0$. So $\tau(A)$ is a measure of how close $A \in \mathcal{M}_n$ to be singular.

Known: $A \circ B^{-1} \in \mathcal{M}_n$ if $A, B \in \mathcal{M}_n$.

Chen (2004) provided a sharper lower bound for $\tau(A \circ B^{-1})$ which improves (7):

$$\tau(A \circ B^{-1}) \geq \tau(A) \tau(B) \min_{i=1,\ldots,n} \left[ \left( \frac{a_{ii}}{\tau(A)} + \frac{b_{ii}}{\tau(B)} - 1 \right) \frac{\beta_{ii}}{b_{ii}} \right].$$

(8)

$$\geq \tau(A) \min_{i=1,\ldots,n} \beta_{ii}$$

(9)

Since $a_{ii} > \tau(A)$ for all $i = 1, \ldots, n$. 
Inequality (8) may be rewritten in the following form:

\[
\rho((A \circ B^{-1})^{-1}) \\
\leq \frac{\rho(A^{-1})\rho(B^{-1})}{\min_{i=1,\ldots,n}[(a_{ii}\rho(A^{-1}) + b_{ii}\rho(B^{-1}) - 1)\frac{\beta_{ii}}{b_{ii}}]}
\]

where \(A \geq 0\) and \(B \in \mathcal{M}_n^{-1}\).

However Chen’s result is not an upper bound for \(\rho(A \circ B^{-1})\).

In view of Corollary 2 and motivated by Chen's bound and its proof, we provide a sharper upper bound for \(\rho(A \circ B)\), where \(A \geq 0\) and \(B \in \mathcal{M}_n^{-1}\).
Theorem 2. Suppose $A \geq 0, B \in \mathcal{M}_n^{-1}$.

1. If $A$ is nilpotent, i.e., $\rho(A) = 0$, then

$$\rho(A \circ B) = 0.$$ 

2. If $A$ is not nilpotent, then

$$\rho(A \circ B) \leq \frac{\rho(A)}{\rho(B)} \max_{i=1,\ldots,n} \left[ \left( \frac{a_{ii}}{\rho(A)} + \beta_{ii} \rho(B) - 1 \right) \frac{b_{ii}}{\beta_{ii}} \right]$$

$$\leq \rho(A) \max_{i=1,\ldots,n} b_{ii}.$$
Remark 2. The first inequality in Theorem 2 is no longer true if we merely assume that $B$ is nonsingular nonnegative. For example, if

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix},$$

and

$$A \circ B = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{pmatrix}, B^{-1} = \frac{1}{9} \begin{pmatrix} 1 & -2 & 4 \\ 4 & 1 & -2 \\ -2 & 4 & 1 \end{pmatrix},$$

then

$$\rho(A) = 1, \quad \rho(B) = 3, \quad \rho(A \circ B) = 2,$$

but

$$\frac{\rho(A)}{\rho(B)} \max_{i=1,\ldots,n} \left[ \left( \frac{a_{ii}}{\rho(A)} + \beta_{ii}(\rho(B) - 1) \right) \frac{b_{ii}}{\beta_{ii}} \right] = -2,$$

not even nonnegative.