Two-species chemotaxis-competition system with singular sensitivity: Global existence, boundedness, and persistence

Halil Ibrahim Kurt *, Wenxian Shen

Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849, USA

Received 19 September 2022; revised 13 January 2023; accepted 21 January 2023
Available online 3 February 2023

Abstract

This paper is concerned with the following parabolic-parabolic-elliptic chemotaxis system with singular sensitivity and Lotka-Volterra competitive kinetics,

\[
\begin{align*}
    u_t &= \Delta u - \chi_1 \nabla \cdot \left( \frac{u}{w} \nabla w \right) + u(a_1 - b_1 u - c_1 v), \\
    v_t &= \Delta v - \chi_2 \nabla \cdot \left( \frac{v}{w} \nabla w \right) + v(a_2 - b_2 v - c_2 u), \\
    0 &= \Delta w - \mu w + \nu u + \lambda v, \\
    \frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, \\
    (0.1)
\end{align*}
\]

where \( \Omega \subseteq \mathbb{R}^N \) is a bounded smooth domain, and \( \chi_i, a_i, b_i, c_i \) \((i = 1, 2)\) and \( \mu, \nu, \lambda \) are positive constants. This is the first work on two-species chemotaxis-competition system with singular sensitivity and Lotka-Volterra competitive kinetics. Among others, we prove that for any given nonnegative initial data \( u_0, v_0 \in C^0(\bar{\Omega}) \) with \( u_0 + v_0 \not\equiv 0 \), (0.1) has a unique globally defined classical solution \( (u(t, x; u_0, v_0), v(t, x; u_0, v_0), w(t, x; u_0, v_0)) \) with \( u(0, x; u_0, v_0) = u_0(x) \) and \( v(0, x; u_0, v_0) = v_0(x) \) in any space dimensional setting with any positive constants \( \chi_i, a_i, b_i, c_i \) \((i = 1, 2)\) and \( \mu, \nu, \lambda \). Moreover, we prove that there is \( \chi^*(\mu, \chi_1, \chi_2) > 0 \) satisfying

\[
\chi^*(\mu, \chi_1, \chi_2) = \begin{cases} 
    \frac{\mu \chi^2}{4} & \text{if } 0 < \chi < 2 \\
    \mu(\chi - 1) & \text{if } \chi \geq 2,
\end{cases}
\]

when \( \chi_1 = \chi_2 :\equiv \chi \)

and

* Corresponding author.
E-mail address: hzk0057@auburn.edu (H.I. Kurt).

https://doi.org/10.1016/j.jde.2023.01.029
0022-0396/© 2023 Elsevier Inc. All rights reserved.
such that the condition

\[ \min\{a_1, a_2\} > \chi^*(\mu, \chi_1, \chi_2) \]

implies

\[ \limsup_{t \to \infty} \|u(t, \cdot; u_0, v_0) + v(t, \cdot; u_0, v_0)\|_{\infty} \leq M^* \text{ and } \liminf_{t \to \infty} \inf_{x \in \Omega} (u(t, x, u_0, v_0) + v(t, x; u_0, v_0)) \geq m^* \]

for some positive constants \( M^*, m^* \) independent of \( u_0, v_0 \), the latter is referred to as combined pointwise persistence.

© 2023 Elsevier Inc. All rights reserved.

MSC: 35K51; 35K57; 35M33; 35Q92; 92C17; 92D25

Keywords: Singular sensitivity; Lotka-Volterra competitive kinetics; Global existence; Global boundedness; Combined mass persistence; Combined pointwise persistence

1. Introduction and main results

Chemotaxis refers to the movement of cells or organisms in response to chemicals in their environments, and plays a crucial role in many biological processes such as immune system response, tumor growth, population dynamics, gravitational collapse, the governing of immune cell migration. Since the pioneering works by Keller and Segel ([24], [25]) on chemotaxis models, a lot of works have been carried out on the qualitative properties of various chemotaxis models such as the analysis of global existence, boundedness, blow-up in finite time, and asymptotic behavior of globally defined solutions, etc. The reader is referred to [4,16,17] and the references therein for some detailed introduction into the mathematics of chemotaxis models.

There are a large number of works on various two competing species chemotaxis models. For example, consider the following two-species chemotaxis system,

\[
\begin{align*}
\frac{u_t}{\Delta u} &= \nabla \cdot (u \chi_1(w) \nabla w) + u(a_1 - b_1 u - c_1 v), & x \in \Omega \\
\frac{v_t}{\Delta v} &= \nabla \cdot (v \chi_2(w) \nabla w) + v(a_2 - b_2 v - c_2 u), & x \in \Omega \\
\tau w_t &= \Delta w - \mu w + \nu u + \lambda v, & x \in \Omega \\
\frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, & x \in \partial \Omega,
\end{align*}
\]

(1.1)

where \( \Omega \subset \mathbb{R}^N \) is a bounded smooth domain, \( a_1, b_1, c_1, a_2, b_2, c_2, \nu, \lambda \) are positive numbers, and \( \tau \geq 0 \). Biologically, (1.1) models the evolution of two competitive species subject to a chemical substance, which is produced by the two species themselves. Here the unknown functions \( u(t, x) \) and \( v(t, x) \) represent the population densities of two competitive biological species and \( w(t, x) \) represents the concentration of the chemical substance. The terms \( u(a_1 - b_1 u - c_1 v) \) and \( v(a_2 - b_2 v - c_2 u) \) are referred to as Lotka-Volterra competitive terms. The parameter \( \mu \) is the degradation rate of the chemical substance and \( \nu \) and \( \lambda \) are the production rates of the chemical substance by the species \( u \) and \( v \), respectively. \( \tau \geq 0 \) is related to the diffusion rate of the
chemical substance. The functions $\chi_1(w)$ and $\chi_2(w)$ reflect the strength of the chemical substance on the movements of two species, and are referred to as chemotaxis sensitivity functions or coefficients.

When $\chi_1(w) \equiv \chi_1 > 0$, $\chi_2(w) \equiv \chi_2 > 0$, and $\tau = 0$, it is known that if $N \leq 2$, or $N \geq 3$ and $\chi_i$ is small relative to $b_i$ and $c_i$ $(i = 1, 2)$, then for any nonnegative initial data $u_0, v_0 \in C^0(\bar{\Omega})$, system (1.1) possesses a unique globally defined classical solution $(u(t, x; u_0, v_0), v(t, x; u_0, v_0), w(t, x; u_0, v_0))$ with $u(0, x; u_0, v_0) = u_0(x)$ and $v(0, x; u_0, v_0) = v_0(x)$ (see [20,21,23,29,33,35] and references therein). Moreover, the large time behaviors of globally defined classical solutions of (1.1) such as competitive exclusion, coexistence, stabilization, etc., are investigated in [7,20,22,23,30,31,33,35], etc.

When $\chi_1(w) \equiv \chi_1 > 0$, $\chi_2(w) \equiv \chi_2 > 0$ and $\tau = 1$, it is proved that, if $N \leq 2$, or $N \geq 3$ and $\chi_1$ and $\chi_2$ are small relative to other parameters in (1.1), then for any nonnegative initial data $u_0, v_0 \in C(\bar{\Omega})$, $w_0 \in W^{1,\infty}(\Omega)$, system (1.1) possesses a unique globally defined classical solution $(u(t, x; u_0, v_0), v(t, x; u_0, v_0), w(t, x; u_0, v_0))$ with $u(0, x; u_0, v_0) = u_0(x)$, $v(0, x; u_0, v_0) = v_0(x)$, and $w(0, x; u_0, v_0) = w_0(x)$ (see [3,29,37], etc.). Moreover, the large-time behaviors of globally defined classical solutions are investigated in [3,37], etc. We refer to the readers to the articles [18,37] for the further details.

The aim of current paper is to investigate the global existence, boundedness, and combined persistence of classical solutions of (1.1) with $\chi_1(w) = \frac{\chi_1}{w}$ and $\chi_2(w) = \frac{\chi_2}{w}$ for some positive constants $\chi_1$ and $\chi_2$, and $\tau = 0$, that is, the following parabolic-parabolic-elliptic chemotaxis system with singular sensitivity and Lotka-Volterra competitive kinetics,

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u - \chi_1 \nabla \cdot \left( \frac{u}{w} \nabla w \right) + u(a_1 - b_1 u - c_1 v), & x \in \Omega \\
\frac{\partial v}{\partial t} &= \Delta v - \chi_2 \nabla \cdot \left( \frac{v}{w} \nabla w \right) + v(a_2 - b_2 v - c_2 u), & x \in \Omega \\
0 &= \Delta w - \mu w + v u + \lambda v, & x \in \Omega \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} &= 0, & x \in \partial \Omega.
\end{align*}
\]

(1.2)

It is seen that the chemotaxis sensitivities $\frac{\chi_i}{w}$ $(i = 1, 2)$ are singular near $w = 0$, reflecting an inhibition of chemotactic migration at high signal concentrations. Such a sensitivity describing the living organisms’ response to the chemical signal was derived by the Weber-Fechner law (see [25]).

We consider classical solutions of (1.2) with initial functions $u_0, v_0 \in C^0(\bar{\Omega})$ with $u_0 \geq 0$, $v_0 \geq 0$, and $\int_{\Omega_2} (u_0 + v_0) > 0$. Note that for such initial functions, if $v_0 = 0$ (resp. $u_0 = 0$), then $v(t, x) \equiv 0$ (resp. $u(t, x) \equiv 0$) on the existence interval. Note also that if $v(t, x) \equiv 0$, then (1.2) becomes

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u - \chi_1 \nabla \cdot \left( \frac{u}{w} \nabla w \right) + u(a_1 - b_1 u), & x \in \Omega \\
0 &= \Delta w - \mu w + v u, & x \in \Omega \\
\frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} &= 0, & x \in \partial \Omega,
\end{align*}
\]

(1.3)

and if $u(t, x) \equiv 0$, then (1.2) becomes

\[
\begin{align*}
\frac{\partial v}{\partial t} &= \Delta v - \chi_2 \nabla \cdot \left( \frac{v}{w} \nabla w \right) + v(a_2 - b_2 v), & x \in \Omega \\
0 &= \Delta w - \mu w + \lambda v, & x \in \Omega \\
\frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} &= 0, & x \in \partial \Omega.
\end{align*}
\]

(1.4)
Systems (1.3) and (1.4) are essentially the same, and are referred to as one species chemotaxis models with logistic source and singular sensitivity. They have been studied in many works (see [5,6,8,13,14,26,27,32], etc.). Let us briefly review some known results for one species chemotaxis models with logistic source and singular sensitivity.

Consider (1.3) with $a_1 = b_1 = 0$ and $\mu = \nu = 1$. Fujie, Winkler, and Yokota in [14] proved the global existence and boundedness of positive classical solutions when $\chi_1 < \frac{2}{N}$ and $N \geq 2$. More recently, Fujie and Senba in [12] proved the global existence and boundedness of classical positive solutions for the case of $N = 2$ for any $\chi_1 > 0$. The existence of finite-time blow-up is then completely ruled out for any $\chi_1 > 0$ in the case $N = 2$. When $N \geq 3$, finite-time blow-up may occur (see [32]).

Consider (1.3) with $a_1, b_1 > 0$. Central questions include whether the logistic source prevents the occurrence of finite-time blow-up in (1.3) (i.e. any positive solution exists globally); if so, whether any globally defined positive solution is bounded, and what is the long time behavior of globally defined bounded positive solutions, etc. When $N = 2$ and $a_1, b_1$ are positive constants, it is proved in [13] that finite-time blow-up does not occur (see [13, Theorem 1.1]), and moreover, if

$$a_1 > \begin{cases} \frac{\mu \chi_1^2}{4}, & \text{if } 0 < \chi_1 \leq 2 \\ \mu (\chi_1 - 1), & \text{if } \chi_1 > 2, \end{cases}$$

(1.5)

then any globally defined positive solution is bounded. Under some additional assumption, it is proved in [8] that the constant solution $\left(\frac{\chi_1}{N}, \frac{\chi_1}{N} \right)$ is exponentially stable (see [8, Theorem 1]). Very recently, among others, we proved that in any space dimensional setting, logistic kinetics prevents the occurrence of finite-time blow-up in (1.3) even for arbitrarily large $\chi_1$ (see [26, Theorem 1.2(1)]). Moreover, under the assumption (1.5), we proved that $\int_{\Omega} u^{-q}$ for some $q > 0$ and $\int_{\Omega} u^p$ for some $p > 2N$ are bounded (see [27, Theorem 1.1]), and that globally defined positive solutions are uniformly bounded and are away from 0 (see [26, Theorem 1.1(2)] and [27, Theorem 1.3]).

However, as far as we know, there is little study on the two-species chemotaxis system (1.2). It is the aim of this paper to investigate the global existence, boundedness, and combined persistence of classical solutions of (1.2).

**Definition 1.1.** For given $u_0(\cdot) \in C^0(\widehat{\Omega})$ and $v_0(\cdot) \in C^0(\widehat{\Omega})$ satisfying that $u_0 \geq 0$, $v_0 \geq 0$, and $\int_{\Omega_1}(u_0(x) + v_0(x))dx > 0$, we say $(u(t, x), v(t, x), w(t, x))$ is a positive classical solution of (1.2) on $(0, T)$ for some $T \in (0, \infty]$ with initial condition $(u(0, x), v(0, x)) = (u_0(x), v_0(x))$ if

$$u(t, x; u_0, v_0) + v(t, x; u_0, v_0) > 0, \quad w(t, x; u_0, v_0) > 0 \quad \forall t \in (0, T), \quad x \in \widehat{\Omega},$$

(1.6)

$$u(\cdot; \cdot), v(\cdot; \cdot) \in C([0, T) \times \widehat{\Omega}) \cap C^{1,2}((0, T) \times \widehat{\Omega}), \quad w(\cdot; \cdot) \in C^{0,2}((0, T) \times \widehat{\Omega}),$$

(1.7)

$$\lim_{t \to 0+} \|u(t, \cdot) - u_0(\cdot)\|_{C^0(\widehat{\Omega})} = 0, \quad \lim_{t \to 0+} \|v(t, \cdot) - v_0(\cdot)\|_{C^0(\widehat{\Omega})} = 0,$$

(1.8)

and $(u(t, x), v(t, x), w(t, x))$ satisfies (1.2) for all $(t, x) \in (0, T) \times \Omega$.

The following proposition on the local existence of classical solutions of (1.2) can be proved by the similar arguments as those in [13, Lemma 2.2].
Proposition 1.1 (Local existence). For given \( u_0(\cdot) \in C^0(\tilde{\Omega}) \) and \( v_0(\cdot) \in C^0(\tilde{\Omega}) \) satisfying that \( u_0 \geq 0, v_0 \geq 0, \) and \( \int_{\Omega}(u_0(x) + v_0(x))dx > 0, \) there exists \( T_{\max}(u_0, v_0) \in (0, \infty] \) such that (1.2) possesses a unique positive classical solution, denoted by \( (u(t, x; u_0, v_0), v(t, x; u_0, v_0), \ w(t, x; u_0, v_0)) \) on \( (0, T_{\max}(u_0, v_0)) \) with initial condition \( (u(0, x; u_0, v_0), v(0, x; u_0, v_0)) = (u_0(x), v_0(x)). \) Moreover, if \( \int_{\Omega}u_0(x)dx > 0 \) and \( \int_{\Omega}v_0(x)dx > 0, \) then

\[
u(t, x; u_0, v_0) > 0 \quad \text{and} \quad \nu(t, x; u_0, v_0) > 0 \quad \forall \ t \in (0, T_{\max}(u_0, v_0)), \ x \in \tilde{\Omega}.
\]

If \( T_{\max}(u_0, v_0) < \infty, \) then either

(1.9)

\[
\limsup_{t \nearrow T_{\max}(u_0, v_0)} \left( \|u(t, \cdot; u_0, v_0) + v(t, \cdot; u_0, v_0)\|_{C^0(\tilde{\Omega})} \right) = \infty,
\]

or

(1.10)

\[
\liminf_{t \nearrow T_{\max}(u_0, v_0)} \inf_{x \in \Omega} w(t, x; u_0, v_0) = 0.
\]

We will focus on the following problems in this paper: whether \( (u(t, x; u_0, v_0), v(t, x; u_0, v_0), \ w(t, x; u_0, v_0)) \) exists globally, i.e., \( T_{\max}(u_0, v_0) = \infty, \) for any \( u_0(\cdot) \in C^0(\tilde{\Omega}) \) and \( v_0(\cdot) \in C^0(\tilde{\Omega}) \) satisfying that \( u_0 \geq 0, v_0 \geq 0, \) and \( \int_{\Omega}(u_0(x) + v_0(x))dx > 0; \) If \( T_{\max}(u_0, v_0) = \infty, \) whether \( (u(t, x; u_0, v_0), v(t, x; u_0, v_0), \ w(t, x; u_0, v_0)) \) is bounded above and stays away from 0.

We point out that in the study of global existence and boundedness of classical solutions of one species chemotaxis model (1.3) with singular sensitivity, it is crucial to prove the boundedness of \( (\int_{\Omega}u(t, x)dx)^{-1} \) and the boundedness of \( \int_{\Omega}u^p(t, x)dx \) for some \( p \gg 1. \) The novel idea discovered in this paper for the study of global existence and boundedness of classical solutions of (1.2) is to prove the boundedness of \( (\int_{\Omega}(u(t, x) + v(t, x))dx)^{-1} \) and the boundedness of \( \int_{\Omega}(u(t, x) + v(t, x))^pdx \) for \( p \gg 1. \) Note that for (1.2), \( (\int_{\Omega}u(t, x)dx)^{-1} \) (resp. \( (\int_{\Omega}v(t, x)dx)^{-1} \)) may not be bounded. Note also that the boundedness of \( (\int_{\Omega}(u(t, x) + v(t, x))dx)^{-1} \) is strongly related to the boundedness of \( \int_{\Omega}(u(t, x) + v(t, x))^{-q}dx \) for some \( q > 0. \)

In the rest of the introduction, we introduce some standing notations in subsection 1.1, and state the main results of the paper and provide some remarks on the main results in subsection 1.2.

1.1. Notations

In this subsection, we introduce some standing notations to be used throughout the paper.

Observe that for given \( u_0(\cdot) \in C^0(\tilde{\Omega}) \) and \( v_0(\cdot) \in C^0(\tilde{\Omega}) \) satisfying that \( u_0 \geq 0, v_0 \geq 0, \) and \( \int_{\Omega}(u_0(x) + v_0(x))dx > 0, \) if \( v_0 \equiv 0 \) (resp. \( u_0 \equiv 0, \) then \( v(t, x; u_0, v_0) \equiv 0 (\text{resp.} \ u(t, x; u_0, v_0) \equiv 0) \) for \( t \in (0, T_{\max}(u_0, v_0)) \) and \( (u(t, x; u_0), w(t, x; u_0)) := (u(t, x; u_0, 0), w(t, x; u_0, 0)) \) (resp. \( v(t, x; v_0), w(t, x; v_0)) := (v(t, x; 0, v_0), w(t, x; 0, v_0)) \) is the solution of (1.3) (resp. (1.4)) with initial condition \( u(0, x; u_0) = u_0(x) \) (resp. \( v(0, x; v_0) = v_0(x) \)). Hence, throughout the rest of this paper, we consider classical solutions of (1.2) with the initial function \( u_0(x), v_0(x) \) satisfying

\[
u_0, v_0 \in C^0(\tilde{\Omega}), \ u_0, v_0 \geq 0, \quad \text{and} \quad \int_{\Omega}u_0 > 0, \int_{\Omega}v_0 > 0.
\]
Let
\[
\begin{align*}
    a_{\min} &= \min\{a_1, a_2\}, \quad a_{\max} = \max\{a_1, a_2\} \\
    b_{\min} &= \min\{b_1, b_2\}, \quad b_{\max} = \max\{b_1, b_2\} \\
    c_{\min} &= \min\{c_1, c_2\}, \quad c_{\max} = \max\{c_1, c_2\}
\end{align*}
\]
(1.12)

For given \(B > 0\) and \(\beta \neq \chi_2 - B\), let
\[
f(\mu, \chi_1, \chi_2, \beta, B) = \mu(B + \beta)\left(1 + \frac{B(\chi_2 - B - \beta)^2 + (\chi_1 - \chi_2)^2\beta}{4B\beta}\right).
\]
(1.13)

For fixed \(\mu > 0, \chi_1 > 0\) and \(\chi_2 > 0\), let
\[
\chi_1^*(\mu, \chi_1, \chi_2) = \inf\left\{f(\mu, \chi_1, \chi_2, \beta, B) \mid B > 0, \beta > 0, \beta \neq \chi_2 - B\right\}.
\]
(1.14)

Similarly, let
\[
\chi_2^*(\mu, \chi_1, \chi_2) = \inf\left\{f(\mu, \chi_2, \chi_1, \beta, B) \mid B > 0, \beta > 0, \beta \neq \chi_1 - B\right\}.
\]
(1.15)

Let
\[
\chi^*(\mu, \chi_1, \chi_2) = \min\{\chi_1^*(\mu, \chi_1, \chi_2), \chi_2^*(\mu, \chi_1, \chi_2)\}.
\]
(1.16)

The number \(\chi^*(\mu, \chi_1, \chi_2)\) will be a lower bound for \(a_{\min}\) for the global boundedness of positive solutions of (1.2). We refer the reader to Lemma 2.1 on the continuity and some specific lower bounds of \(\chi^*(\mu, \chi_1, \chi_2)\).

1.2. Main results and remarks

In this subsection, we state the main results of the paper and provide some remarks on the main results.

Observe that, for any given \(u_0, v_0\) satisfying (1.11), by the third equation in (1.2),
\[
\mu \int_{\Omega} w(t, x; u_0, v_0) dx = \int_{\Omega} (vu(t, x; u_0, v_0) + \lambda v(t, x; u_0, v_0)) dx
\]
and
\[
\min\{v, \lambda\} \int_{\Omega} (u(t, x; u_0, v_0) + v(t, x; u_0, v_0)) dx \leq \int_{\Omega} (vu(t, x; u_0, v_0) + \lambda v(t, x; u_0, v_0)) dx
\]
\[
\quad \leq \max\{v, \lambda\} \int_{\Omega} (u(t, x; u_0, v_0) + v(t, x; u_0, v_0)) dx
\]
for any \(t \in (0, T_{\max}(u_0, v_0))\). Hence \(\liminf_{t \to T_{\max}(u_0, v_0)} \int_{\Omega} (u(t, x; u_0, v_0) + v(t, x; u_0, v_0)) dx = 0\) implies that (1.10) holds. On the other hand, we have

253
\[ w(t, x; u_0, v_0) \geq \delta_0 \int_{\Omega} (u(t, x; u_0, v_0) + v(t, x; u_0, v_0)) > 0 \quad \forall x \in \Omega, \; t \in (0, T_{\max}(u_0, v_0)) \]

for some positive constant \( \delta_0 \) independent of \( u_0, v_0 \) (see Lemma 2.4). Hence

\[
\liminf_{t \nearrow T_{\max}(u_0, v_0)} \inf_{x \in \Omega} w(t, x; u_0, v_0) = 0 \iff \liminf_{t \nearrow T_{\max}(u_0, v_0)} \int_{\Omega} (u(t, x; u_0, v_0) + v(t, x; u_0, v_0)) dx = 0.
\]

The first main theorem of the current paper is on the lower bounds of the combined mass
\[
\int_{\Omega} (u(t, x; u_0, v_0) + v(t, x; u_0, v_0)) \, dx
\]
on any bounded subinterval of \((0, T_{\max}(u_0, v_0))\), which would provide the upper bounds of \( (\int_{\Omega} (u(t, x; u_0, v_0) + v(t, x; u_0, v_0)) \, dx)^{-1} \) on bounded subintervals of \((0, T_{\max}(u_0, v_0))\).

**Theorem 1.1 (Local lower bound of the combined mass).** For any \( T \in (0, \infty) \) and any \( u_0, v_0 \) satisfying (1.11),

\[
\inf_{0 \leq t < \min(T, T_{\max}(u_0, v_0))} \int_{\Omega} (u(t, x; u_0, v_0) + v(t, x; u_0, v_0)) dx > 0. \tag{1.18}
\]

**Remark 1.1.**

1. By Proposition 1.1, (1.17) and (1.18), if \( T_{\max}(u_0, v_0) < \infty \), then we must have

\[
\limsup_{t \nearrow T_{\max}(u_0, v_0)} \left( \|u(t, \cdot; u_0, v_0) + v(t, \cdot; u_0, v_0)\|_{C^0(\Omega)} \right) = \infty.
\]

2. If

\[
\inf_{0 \leq t < \min(T, T_{\max}(u_0, v_0))} \int_{\Omega} u(t, x; u_0, v_0) dx > 0 \tag{1.19}
\]

and

\[
\inf_{0 \leq t < \min(T, T_{\max}(u_0, v_0))} \int_{\Omega} v(t, x; u_0, v_0) dx > 0, \tag{1.20}
\]

then (1.18) holds. But the converse may not be true because competitive exclusion may occur in (1.2), which will be studied somewhere else.

3. It will be proved that \( T_{\max}(u_0, v_0) = \infty \) for any \( u_0, v_0 \) satisfying (1.11) (see Theorem 1.3). It will be also proved that under the assumption \( a_{\inf} > \chi^*(\mu, \chi_1, \chi_2) \), there is \( q > 0 \) such that

\[
\limsup_{t \to \infty} \int_{\Omega} (u(t, x; u_0, v_0) + v(t, x; u_0, v_0))^{-q} dx < \infty
\]

(see Theorem 1.4(1)). Note that
\[ \int_{\Omega} (u + v) \geq |\Omega|^{q+1 \over q} \left( \int_{\Omega} (u + v)^{-q} \right)^{-{1 \over q}}. \]

Hence, under the assumption \( a_{\inf} > \chi^*(\mu, \chi_1, \chi_2) \), (1.18) holds with \( T = \infty \), which is referred to as combined mass persistence (see also Remark 1.5(2)). It should be pointed out that mass persistence or persistence of mass was first addressed by Tao and Winkler in [34] for the following chemotaxis system,

\[
\begin{cases}
    u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + ru - \mu u^2, & x \in \Omega \\
    v_t = \Delta v - v + u, & x \in \Omega \\
    \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial \Omega
\end{cases}
\]

(see [34, Theorem 1.1]). Our mass persistence result for (1.2) is an analogue of [34, Theorem 1.1]. Besides the mass persistence, we also prove the pointwise persistence for the solutions of (1.2) (see Theorem 1.5(2)), which is stronger than the mass persistence.

The second main theorem of the current paper is on the local \( L^p \)- and \( C^\theta \)-boundedness of positive classical solutions of (1.2).

**Theorem 1.2 (Local \( L^p \)- and \( C^\theta \)-boundedness).**

1. **(Local \( L^p \)-boundedness)** There are \( \varepsilon_0 > 0 \), \( p > 3N \) and \( M_1 > 0 \) such that for any \( T \in (0, \infty) \) and any \( u_0, v_0 \) satisfying (1.11),

\[
\int_{\Omega} (u + v)^p (t, x; u_0, v_0) \, dx \leq e^{-(t - \hat{T})} \int_{\Omega} (u + v)^p (\tau, x; u_0, v_0) \, dx
\]

\[
+ M_1 \left( \frac{1}{\inf_{\tau \leq t < \hat{T}, x \in \Omega} w^{p+1 - \varepsilon_0 / 2}(\tau, x; u_0, v_0)} + 1 \right)^{2(p+1)/\varepsilon_0} + M_1
\]

and

\[
\int_{\Omega} (u + v)^p (t, x; u_0, v_0) \, dx \leq \max \left\{ e^{-(t - \tau)} \int_{\Omega} (u + v)^p (\tau, x; u_0, v_0) \, dx, \right. \\
\left. M_1 \left( \frac{1}{\inf_{\tau \leq t < \hat{T}, x \in \Omega} w^{p+1 - \varepsilon_0 / 2}(\tau, x; u_0, v_0)} + 1 \right)^{2(p+1)/\varepsilon_0} + M_1 \right\}
\]

for any \( 0 \leq \tau \leq t < \hat{T} \), where \( \hat{T} = \min\{T, T_{\max}(u_0, v_0)\} \).

2. **(Local \( C^\theta \)-boundedness)** For any \( p > 2N \) and \( 0 < \theta < 1 - {2N \over p} \), there are \( M_2 > 0 \), \( \beta > 0 \), and \( \gamma > 0 \) such that for any \( u_0, v_0 \) satisfying (1.11) and \( T \in (0, \infty) \),

\[
\| u(t, \cdot; u_0, v_0) + v(t, \cdot; u_0, v_0) \|_{C^\theta(\Omega)}
\]

\[
\leq M_2 \left[ (t - \tau)^{-\beta} e^{-\gamma(t - \tau)} \| u(\tau, \cdot; u_0, v_0) + v(\tau, \cdot; u_0, v_0) \|_{L^p} \right]
\]

255
\[
\begin{align*}
\sup_{\tau \leq t < \hat{T}} \|u(t, \cdot; u_0, v_0) + v(t, \cdot; u_0, v_0)\|_{L^p}^2 \\
+ \inf_{\tau \leq t < \hat{T}, x \in \Omega} w(t, x; u_0, v_0) \\
+ \sup_{\tau \leq t < \hat{T}} \|u(t, \cdot; u_0, v_0) + v(t, \cdot; u_0, v_0)\|_{L^p} + 1
\end{align*}
\]

for any \(0 \leq \tau < \hat{T} = \min\{T, T_{\max}(u_0, v_0)\}\) and \(\tau < t < \hat{T}\).

**Remark 1.2.**

1. By Theorem 1.2, if \(\inf_{x \in \Omega} w(t, x; u_0, v_0)\) is bounded away from zero on \((0, T_{\max}(u_0, v_0))\), then \(\int_{\Omega}(u^p(t, x; u_0, v_0) + v^p(t, x; u_0, v_0))dx\) is bounded on \((0, T_{\max}(u_0, v_0))\) for some \(p > 3N\) and then \(\|u(t, \cdot; u_0, v_0) + v(t, \cdot; u_0, v_0)\|_{C^0(\hat{\Omega})}\) is bounded on \([\tau, T_{\max}(u_0, v_0)\) for some \(0 < \theta < 1\) and any \(0 < \tau < T_{\max}(u_0, v_0)\).

2. By Theorems 1.1 and 1.2, \(\|u(t, \cdot; u_0, v_0) + v(t, \cdot; u_0, v_0)\|_{C^0(\hat{\Omega})}\) is bounded on \([\tau, \hat{T})\) for some \(\theta \in (0, 1)\), any \(T \in (0, \infty)\), and any \(0 < \tau < \hat{T} := \min\{T, T_{\max}(u_0, v_0)\}\), which plays an important role in the study of global existence of classical solutions of (1.2).

The third main theorem of the current paper is on global existence of classical solutions of (1.2).

**Theorem 1.3 (Global existence).** For any \(u_0\) and \(v_0\) satisfying (1.11), \((u(t, x; u_0, v_0), v(t, x; u_0, v_0), w(t, x; u_0, v_0))\) exists globally, that is, \(T_{\max}(u_0, v_0) = \infty\).

**Remark 1.3.**

1. By (1.17), (1.18), and Theorem 1.1, for any \(u_0, v_0\) satisfying (1.11),

\[
\inf_{0 \leq t < T} \inf_{x \in \Omega} w(t, x; u_0, v_0) \geq \delta_0 \inf_{0 \leq t < T} \int_\Omega (u(t, x; u_0, v_0) + v(t, x; u_0, v_0))dx > 0
\]

\[\forall 0 < T < \infty.\] (1.22)

2. By (1.22) and Theorem 1.2,

\[
\sup_{t \in (0, T)} \|u(t, \cdot; u_0, v_0) + v(t, \cdot; u_0, v_0)\|_{L^\infty(\Omega)} < \infty \quad \forall 0 < T < \infty.\] (1.23)

3. It is interesting to see whether (1.22) and (1.23) also hold for \(T = \infty\), which is strongly related to the boundedness of \(\int_{\Omega}(u^q(t, x; u_0, v_0) + v^q(t, x; u_0, v_0))dx\) for some \(q > 0\) and the boundedness of \(\int_{\Omega}(u(t, x; u_0, v_0) + v(t, x; u_0, v_0))^pdx\) for some \(p > 3N\), and is discussed in Theorems 1.4 and 1.5 in the following.

The fourth main theorem of the current paper is on the global boundedness of \(\int_{\Omega}(u + v)^{-q}\) and \(\int_{\Omega}(u + v)^p\) for some \(q > 0\) and \(p > 3N\). For given \(u_0, v_0\) satisfying (1.11) and \(\tau \geq 0\), let
\[
    m^*(\tau, u_0, v_0) = \max\left\{ \int_\Omega u(\tau, x; u_0, v_0)dx, \frac{a_1|\Omega|}{b_1} \right\} + \max\left\{ \int_\Omega v(\tau, x; u_0, v_0)dx, \frac{a_2|\Omega|}{b_2} \right\}.
\]

(1.24)

**Theorem 1.4.** Assume \( a_{\min} > \chi^*(\mu, \chi_1, \chi_2) \). Then the following hold.

1. **(Boundedness of \( \int_\Omega (u + v)^q \))** There are \( q > 0 \) and \( M_3 > 0 \) such that for any \( u_0, v_0 \) satisfying (1.11) and for every \( 0 < \tau < \infty \),

\[
    \int_\Omega (u(t, x; u_0, v_0) + v(t, x; u_0, v_0))^{-q} dx \\
    \leq e^{-\frac{\epsilon_0 q}{2}(t-\tau)} \int_\Omega (u(\tau, x; u_0, v_0) + v(\tau, x; u_0, v_0))^{-q} dx + 2qC_{\tau, u_0, v_0}\epsilon_0^{-1} + M_3
\]

for all \( t > \tau \), where \( \epsilon_0 = a_{\min} - \chi^*(\mu, \chi_1, \chi_2) \) and

\[
    C_{\tau, u_0, v_0} = \begin{cases} 
        0 & \text{if } q \geq 1, \\
        (b_{\max} + c_{\max})|\Omega|^q(m^*(\tau, u_0, v_0))^{1-q} & \text{if } q < 1.
    \end{cases}
\]

2. **(Boundedness of \( \int_\Omega (u + v)^p \))** Let \( p > 3N \) and \( \epsilon_0 > 0 \) be as in Theorem 1.2(1). Then there is \( M_4 > 0 \) such that for every \( \tau > 0 \),

\[
    \int_\Omega (u + v)^p(t, x; u_0, v_0)dx \leq e^{-(t-\tau)} \int_\Omega (u + v)^p(\tau, x; u_0, v_0)dx \\
    + M_4\left(\inf_{\tau \leq t < \infty, x \in \Omega} \frac{1}{w^{p+1-\epsilon_0/2}(t, x; u_0, v_0)} + 1\right)^{2(p+1)/\epsilon_0} + M_4
\]

for all \( t > \tau \).

**Remark 1.4.** The concrete estimates for \( \int_\Omega (u(t, x; u_0, v_0) + v(t, x; u_0, v_0))^{-q} dx \) and \( \int_\Omega (u(t, x; u_0, v_0) + v(t, x; u_0, v_0))^{p} dx \) obtained in Theorem 1.4 provide useful tools for the study of the asymptotic behavior of globally defined positive solutions of (1.2).

The last main theorem of the current paper is on uniform boundedness and uniform pointwise persistence of classical solutions of (1.2), which provides some insight into the understanding of the asymptotic behavior of globally defined positive solutions of (1.2).

**Theorem 1.5.** Assume \( a_{\min} > \chi^*(\mu, \chi_1, \chi_2) \). Then the following hold.

1. **(Uniform boundedness)** Let \( p > 3N \) be as in Theorem 1.2(1) and \( q > 0 \) be as in Theorem 1.4(1). Let \( 0 < \theta < \frac{N}{2p} \). There are \( M_1^* > 0, M_2^* > 0 \) and \( M_3^* > 0 \) such that for any \( u_0, v_0 \) satisfying (1.11),
\[
\limsup_{t \to \infty} \int_{\Omega} (u(t, x; u_0, v_0) + v(t, x; u_0, v_0))^{-q} \, dx \leq M_1^*, \quad (1.25)
\]

\[
\limsup_{t \to \infty} \int_{\Omega} (u(t, x; u_0, v_0) + v(t, x; u_0, v_0))^p \, dx \leq M_2^*, \quad (1.26)
\]

and

\[
\limsup_{t \to \infty} \|u(t, \cdot; u_0, v_0) + v(t, \cdot; u_0, v_0)\|_{C^{0}(\bar{\Omega})} \leq M_3^*. \quad (1.27)
\]

(2) (Combined pointwise persistence) There is \( M_0^* > 0 \) such that for any \( u_0, v_0 \) satisfying (1.11),

\[
\liminf_{t \to \infty} \inf_{x \in \Omega} (u(t, x; u_0, v_0) + v(t, x; u_0, v_0)) \geq M_0^*. \quad (1.28)
\]

**Remark 1.5.**

(1) Let

\[
\mathcal{E} = \left\{ u, v \in C^0(\bar{\Omega}) \mid u \geq 0, \; v \geq 0, \; \int_{\Omega} (u(x) + v(x)) \, dx > 0, \; \int_{\Omega} (u(x) + v(x))^{-q} \, dx \leq M_1^*, \; \int_{\Omega} (u(x) + v(x))^p \, dx \leq M_2^* \right\}.
\]

Theorem 1.4 shows that \( \mathcal{E} \) eventually attracts every globally defined positive solution of (1.2).

(2) (1.28) implies that if \( a_{\min} > \chi^*(\mu, \chi_1, \chi_2) \), then there is \( m_0^* > 0 \) such that for any \( u_0, v_0 \) satisfying (1.11),

\[
\liminf_{t \to \infty} \int_{\Omega} (u(t, x; u_0, v_0) + v(t, x; u_0, v_0)) \, dx \geq m_0^*, \quad (1.29)
\]

which is referred to as combined mass persistence. It remains open whether (1.29) holds without the assumption \( a_{\min} > \chi^*(\mu, \chi_1, \chi_2) \).

(3) (1.27) implies that if \( a_{\min} > \chi^*(\mu, \chi_1, \chi_2) \), then for any \( u_0, v_0 \) satisfying (1.11),

\[
\limsup_{t \to \infty} \|u(t, \cdot; u_0, v_0) + v(t, \cdot; u_0, v_0)\|_{L^\infty(\Omega)} \leq M_3^*. \quad (1.30)
\]

It also remains open whether (1.30) holds without the assumption \( a_{\min} > \chi^*(\mu, \chi_1, \chi_2) \).

(4) We say both species persistent in mass if for any \( u_0, v_0 \) satisfying (1.11),

\[
\liminf_{t \to \infty} \int_{\Omega} u(t, x; u_0, v_0) \, dx > 0 \quad \text{and} \quad \liminf_{t \to \infty} \int_{\Omega} v(t, x; u_0, v_0) \, dx > 0,
\]

or
and say competitive exclusion in mass occurs if for any \(u_0, v_0\) satisfying (1.11),

\[
\limsup_{t \to \infty} \int_{\Omega} u(t, x; u_0, v_0) dx = 0 \quad \text{and} \quad \liminf_{t \to \infty} \int_{\Omega} v(t, x; u_0, v_0) dx > 0
\]

or

\[
\liminf_{t \to \infty} \int_{\Omega} u(t, x; u_0, v_0) dx > 0 \quad \text{and} \quad \limsup_{t \to \infty} \int_{\Omega} v(t, x; u_0, v_0) dx = 0.
\]

We say both species persistent pointwise if for any \(u_0, v_0\) satisfying (1.11),

\[
\liminf_{t \to \infty} \inf_{x \in \Omega} u(t, x; u_0, v_0) > 0 \quad \text{and} \quad \liminf_{t \to \infty} \inf_{x \in \Omega} v(t, x; u_0, v_0) > 0,
\]

and say pointwise competitive exclusion occurs if for any \(u_0, v_0\) satisfying (1.11),

\[
\limsup_{t \to \infty} \sup_{x \in \Omega} u(t, x; u_0, v_0) = 0 \quad \text{and} \quad \liminf_{t \to \infty} \inf_{x \in \Omega} v(t, x; u_0, v_0) > 0
\]

or

\[
\liminf_{t \to \infty} \inf_{x \in \Omega} u(t, x; u_0, v_0) > 0 \quad \text{and} \quad \limsup_{t \to \infty} \sup_{x \in \Omega} v(t, x; u_0, v_0) = 0.
\]

We will study the persistence of both species and competitive exclusion somewhere else.

We conclude the introduction with some remarks on the following full parabolic counterpart of (1.2),

\[
\begin{align*}
  u_t &= Δu - χ_1 \nabla \cdot (\frac{K_1}{w} \nabla w) + u(a_1 - b_1 u - c_1 v), \quad x \in \Omega \\
  v_t &= Δv - χ_2 \nabla \cdot (\frac{K_2}{w} \nabla w) + v(a_2 - b_2 v - c_2 u), \quad x \in \Omega \\
  w_t &= Δw - λu + νu + λv, \quad x \in \Omega \\
  \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} &= 0, \quad x \in \partial \Omega.
\end{align*}
\]

(1.31)

For given \(u_0, v_0, w_0\) satisfying

\[
u_0, v_0(·) \in C^0(\bar{Ω}), \ w_0 \in W^{1,∞}(Ω), \ u_0 \geq 0, \ v_0 \geq 0, \ \int_{Ω} (u_0(x) + v_0(x)) dx > 0, \ w_0(x) > 0, \]

(1.32)

by standard contraction arguments (see [4,13,19]), there exists \(T_{max}(u_0, v_0, w_0) \in (0, \infty]\) such that (1.31) possesses a unique positive classical solution, denoted by \((u(t, x; u_0, v_0, w_0), v(t, x; u_0, v_0, w_0), w(t, x; u_0, v_0, w_0))\), on \((0, T_{max}(u_0, v_0, w_0))\) with initial condition \((u(0, x; u_0, v_0, w_0)), v(0, x; u_0, v_0, w_0), w(0, x; u_0, v_0, w_0)) = (u_0(x), v_0(x), w_0(x))\). Moreover if \(T_{max}(u_0, v_0, w_0) < \infty\), then either
\[
\limsup_{t \to T_{\max}(u_0, v_0, w_0)} \left( \|u(t, \cdot; u_0, v_0, w_0) + v(t, \cdot; u_0, v_0, w_0)\|_{C^0(\bar{\Omega})} + \|w(t, \cdot; u_0, v_0, w_0)\|_{W^{1,\infty}(\Omega)} \right) = \infty,
\]
or
\[
\liminf_{t \to T_{\max}(u_0, v_0, w_0)} \inf_{x \in \Omega} w(t, x; u_0, v_0, w_0) = 0.
\]

Observe that, for given \(u_0, v_0, w_0\) satisfying (1.32), if \(v_0(x) \equiv 0\), then \(v(t, x; u_0, v_0, w_0) \equiv 0\) for \(t \in (0, T_{\max}(u_0, v_0, w_0))\) and \((u(t, x; u_0, w_0), w(t, x; u_0, w_0)) := (u(t, x; u_0, v_0, w_0), w(t, x; u_0, v_0, w_0))\) is the classical solution of the following full parabolic counterpart of (1.3)

\[
\begin{cases}
 u_t = \Delta u - \chi_1 \nabla \cdot (\frac{u}{w} \nabla w) + u(a_1 - b_1 u), & x \in \Omega \\
 w_t = \Delta w - \mu w + \nu u, & x \in \Omega \\
 \frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0, & x \in \partial \Omega
\end{cases} \tag{1.33}
\]

with initial condition \((u(0, x; u_0, w_0), w(0, x; u_0, w_0)) = (u_0(x), w_0(x))\). Similarly, if \(u_0 \equiv 0\), then \(u(t, x; u_0, v_0, w_0) \equiv 0\) on \((0, T_{\max}(u_0, v_0, w_0))\) and \((v(t, x; v_0, w_0), w(t, x; v_0, w_0)) := (v(t, x; u_0, v_0, w_0), w(t, x; u_0, v_0, w_0))\) is the classical solution of

\[
\begin{cases}
 v_t = \Delta v - \chi_2 \nabla \cdot (\frac{v}{w} \nabla w) + v(a_2 - b_2 v), & x \in \Omega \\
 w_t = \Delta w - \mu w + \lambda v, & x \in \Omega \\
 \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, & x \in \partial \Omega
\end{cases} \tag{1.34}
\]

with initial condition \((v(0, x; v_0, w_0), w(0, x; v_0, w_0)) = (v_0(x), w_0(x))\).

Systems (1.33) and (1.34) are essentially the same. There are several works on the global existence and asymptotic behavior of classical or weak solutions of (1.33). For example, in the case \(N = 2\), the authors of [1] proved the global existence of solutions of (1.33) with initial functions \(u_0 \in L^2(\Omega), u_0 \geq 0, \) and \(w_0 \in H^{1+\theta_0}(\Omega)\) for some \(\theta_0 \in (0, 1/2)\) satisfying \(\text{inf}_{x \in \Omega} w_0(x) > 0\) (see [1, Theorem 2.1]); the authors of [38] proved the global existence and boundedness of classical solutions of (1.33) with initial functions \(u_0 \in C^0(\Omega), u_0(x) \geq 0, u_0 \not\equiv 0, \) and \(w_0 \in W^{1,q}(\Omega)\) for some \(q > 2\) provided that \(\chi_1\) is relatively small with respect to \(a_1\) (see [38, Theorem 1]); the authors of [40] showed the global stability of the positive constant solution of (1.33) provided that \(\chi_1\) is relatively small with respect to \(a_1\) (see [40, Theorem 1.1]). For general \(N \geq 1\), global existence of weak solutions of (1.33) is studied in [10,39]. The authors of [11,28,36] studied the global existence and boundedness of classical solutions of (1.33) with \(a_1 = b_1 = 0\).

Up to our knowledge, it remains open whether (1.33) has a global classical solution for any given initial functions \(u_0 \in C^0(\Omega), u_0(x) \geq 0, u_0 \not\equiv 0, \) and \(w_0 \in W^{1,\infty}(\Omega), w_0(x) > 0\) in any space dimensional setting. There is little study on the global existence and boundedness of classical solutions of (1.31) for any given initial functions \(u_0, v_0, w_0\) satisfying (1.32). We remark that finite upper bounds of \(\int_{\Omega} (u(t, x) + v(t, x)) p \, dx\) for some \(p \gg 1\) and positive lower bounds of \(w(t, x)\), or equivalently, finite upper bounds of \(\int_{\Omega} (u(t, x) + v(t, x))^{-q} \, dx\) for some \(q > 0\), are among the key ingredients in the proofs of global existence and boundedness of the classical solution \(u(t, x), v(t, x), w(t, x))\) of (1.2) with initial functions \(u_0, v_0\) satisfying (1.11). We expect that
such bounds for the solutions of (1.31) if they can be obtained will also ensure the global existence and boundedness of classical solutions of (1.31) with initial conditions $u_0$, $v_0$, $w_0$ satisfying (1.32). However, the techniques in the current paper to obtain the upper bounds of $\int_{\Omega}(u + v)^p$ for $p \gg 1$ and $\int_{\Omega}(u + v)^{-q}$ for some $q > 0$ for the solutions of (1.2) rely on the fact that the equation for $u$ in (1.2) is elliptic (see Remarks 3.2 and 4.1). New techniques/methods need to be developed to get such bounds for the solutions of (1.31). We wish to carry out some study on the global existence and various properties of classical solutions of (1.31) in the near future.

The rest of the paper is organized as follows. In section 2, we present some preliminary lemmas. In section 3, we study local lower bound of $\int_{\Omega}(u(t, x) + v(t, x))dx$; local upper bounds of $\int_{\Omega}(u(t, x) + v(t, x))^p dx$ and $\|u(t, \cdot) + v(t, \cdot)\|_{C^0(\overline{\Omega})}$ for some $p > 3N$ and $\theta \in (0, 1)$; the global existence of classical solutions of (1.2), and prove Theorems 1.1-1.3. We investigate the global boundedness of $\int_{\Omega}(u + v)^{-q}$ and $\int_{\Omega}(u + v)^p$ and prove Theorems 1.4 and 1.5 in section 4. We prove two important technical propositions in the Appendix.

2. Preliminary lemmas

In this section, we present some lemmas to be used in later sections.

First, let $\chi^*(\mu, \chi_1, \chi_2)$ be defined as in (1.16). We present the following lemma on the continuity and lower bounds of $\chi^*(\mu, \chi_1, \chi_2)$.

**Lemma 2.1.**

1. $\chi^*(\mu, \chi_1, \chi_2)$ is upper semicontinuous in $\mu > 0$, $\chi_1 > 0$ and $\chi_2 > 0$, that is,

$$\limsup_{(\mu, \chi_1, \chi_2) \to (\mu_0, \chi_1^0, \chi_2^0)} \chi^*(\mu, \chi_1, \chi_2) \leq \chi^*(\mu_0, \chi_1^0, \chi_2^0) \quad \forall \mu_0 > 0, \chi_1^0 > 0, \chi_2^0 > 0.$$

2. For any $\mu > 0$, $\chi_1 > 0$ and $\chi_2 > 0$,

$$\chi^*(\mu, \chi_1, \chi_2) \leq \min\left\{\mu\chi_2 + \frac{\mu(\chi_1 - \chi_2)^2}{4}, \mu\chi_1 + \frac{\mu(\chi_2 - \chi_1)^2}{4}\right\}, \quad (2.1)$$

and when $\chi_1 = \chi_2 = \chi$,

$$\chi^*(\mu, \chi, \chi) \leq \begin{cases} \frac{\mu\chi^2}{4} & \text{if } 0 < \chi < 2 \\ \mu(\chi - 1) & \text{if } \chi \geq 2. \end{cases} \quad (2.2)$$

**Remark 2.1.** We point out that the boundedness of $\int_{\Omega}(u(t, x; u_0, v_0) + v(t, x; u_0, v_0))^q dx$ for some $q > 0$ plays a crucial role in the study of the boundedness of $u(t, x; u_0, v_0) + v(t, x; u_0, v_0)$. The assumption that $a_{\text{min}} > \chi^*(\mu, \chi_1, \chi_2)$ indicates that the chemotaxis sensitivities $\chi_1, \chi_2$ and the degradation rate $\mu$ of the chemical substance are small relatively with respect to $a_{\text{min}}$ and ensures the boundedness of $\int_{\Omega}(u(t, x; u_0, v_0) + v(t, x; u_0, v_0))^q dx$ for some $q > 0$ (see Lemma 4.1 and the arguments of Theorem 1.4(1)). In the following, we provide some discussions on some specific upper bounds of $\chi^*(\mu, \chi_1, \chi_2)$ for the reader to get some comprehensive feeling about the impact of $\chi_1, \chi_2$ and $\mu$ on the boundedness of $\int_{\Omega}(u + v)^q$. First, note that
\[
\frac{\partial}{\partial t} (u + v) = \Delta (u + v) - \nabla \cdot \left( \frac{\chi_1 u + \chi_2 v}{w} \nabla w \right) \\
+ (a_1 u + a_2 v) - (b_1 u^2 + b_2 v^2) - (c_1 + c_2)uv.
\] (2.3)

Next, for any \( q > 0 \), multiplying (2.3) by \( (u + v)^{-q-1} \) and integrating over \( \Omega \), we have

\[
\frac{1}{q} \frac{d}{dt} \int_{\Omega} (u + v)^{-q} \\
= -(q + 1) \int_{\Omega} (u + v)^{-q-2} |\nabla (u + v)|^2 \\
+ (q + 1) \int_{\Omega} (u + v)^{-q-2} \frac{\chi_1 u + \chi_2 v}{w} \nabla (u + v) \cdot \nabla w \\
- \int_{\Omega} (u + v)^{-q-1} (a_1 u + a_2 v) + \int_{\Omega} (u + v)^{-q-1} (b_1 u^2 + b_2 v^2) \\
+ \int_{\Omega} (u + v)^{-q-1} (c_1 + c_2)uv \\
\leq -(q + 1) \int_{\Omega} (u + v)^{-q-2} |\nabla (u + v)|^2 \\
+ (q + 1) \chi_2 \int_{\Omega} \frac{(u + v)^{-q-1}}{w} \nabla (u + v) \cdot \nabla w \\
+ (q + 1)(\chi_1 - \chi_2) \int_{\Omega} (u + v)^{-q-2} \frac{u}{w} \nabla (u + v) \cdot \nabla w \\
- a_{\min} \int_{\Omega} (u + v)^{-q} + b_{\max} \int_{\Omega} (u + v)^{-q+1} + c_{\max} \int_{\Omega} (u + v)^{-q+1} \\
\leq -(q + 1) \int_{\Omega} (u + v)^{-q-2} |\nabla (u + v)|^2 \\
+ (q + 1) \chi_2 \int_{\Omega} \frac{(u + v)^{-q-1}}{w} \nabla (u + v) \cdot \nabla w \\
+ (\chi_1 - \chi_2)^2 q (q + 1) \int_{\Omega} (u + v)^{-q-2} |\nabla (u + v)|^2 + B(q + 1) \int_{\Omega} (u + v)^{-q} \frac{\|\nabla w\|^2}{w^2} \\
- a_{\min} \int_{\Omega} (u + v)^{-q} + (b_{\max} + c_{\max}) \int_{\Omega} (u + v)^{-q+1}, \ \forall B > 0.
\] (2.4)
Now, by the third equation in (1.2), we have

\[
\frac{B}{q} \int_{\Omega} (u + v)^{-q} \frac{|\nabla w|^2}{w^2} \leq \frac{B\mu}{q} \int_{\Omega} (u + v)^{-q} - B \int_{\Omega} \frac{(u + v)^{-q - 1}}{w} \nabla(u + v) \cdot \nabla w
\]

(see the arguments of Lemma 4.1 for detail). Hence

\[
\frac{1}{q} \frac{d}{dt} \int_{\Omega} (u + v)^{-q} \leq - (q + 1) \int_{\Omega} (u + v)^{-q - 2} |\nabla(u + v)|^2
\]

\[
+ (q + 1) \chi_2 \int_{\Omega} \frac{(u + v)^{-q - 1}}{w} \nabla(u + v) \cdot \nabla w
\]

\[
+ \frac{(\chi_1 - \chi_2)^2 q(q + 1)}{4B} \int_{\Omega} (u + v)^{-q - 2} |\nabla(u + v)|^2
\]

\[
+ \frac{B(q + 1)\mu}{q} \int_{\Omega} (u + v)^{-q} - B(q + 1) \int_{\Omega} \frac{(u + v)^{-q - 1}}{w} \nabla(u + v) \cdot \nabla w
\]

\[
- a_{\min} \int_{\Omega} (u + v)^{-q} + (b_{\max} + c_{\max}) \int_{\Omega} (u + v)^{-q + 1}
\]

for any \( q > 0, B > 0 \). Let \( B = \chi_2 \) and \( q = \frac{4\chi_2}{(\chi_1 - \chi_2)^2} \) in the case \( \chi_1 \neq \chi_2 \). We get

\[
\frac{1}{q} \frac{d}{dt} \int_{\Omega} (u + v)^{-q} \leq - \left( a_{\min} - \mu \left( \chi_2 + \frac{(\chi_1 - \chi_2)^2}{4} \right) \right) \int_{\Omega} (u + v)^{-q} + (b_{\max} + c_{\max}) \int_{\Omega} (u + v)^{-q + 1}.
\]

Therefore, \( a_{\min} > \mu \left( \chi_2 + \frac{(\chi_1 - \chi_2)^2}{4} \right) \) ensures the boundedness of \( \int_{\Omega} (u + v)^{-q} \) (see the arguments of Theorem 1.4(1)) and \( \mu \left( \chi_2 + \frac{(\chi_1 - \chi_2)^2}{4} \right) \) is an upper bound of \( \chi^*(\mu, \chi_1, \chi_2) \). Similarly, \( \mu \left( \chi_1 + \frac{(\chi_1 - \chi_2)^2}{4} \right) \) is an upper bound of \( \chi^*(\mu, \chi_1, \chi_2) \).

**Proof of Lemma 2.1.** (1) We prove that \( \chi^*(\mu, \chi_1, \chi_2) \) is upper semicontinuous in \( \mu > 0, \chi_1 > 0 \) and \( \chi_2 > 0 \).

Fix \( (\mu^0, \chi_1^0, \chi_2^0) \) with \( \mu^0, \chi_1^0, \chi_2^0 > 0 \). Without loss of generality, we may assume that \( \chi^*(\mu^0, \chi_1^0, \chi_2^0) = \chi_1^*(\mu^0, \chi_1^0, \chi_2^0) \). Then, for any \( \epsilon > 0 \), there are \( \bar{\beta} > 0 \) and \( \bar{B} > 0 \) satisfying \( \bar{\beta} \neq \chi_2 - \bar{B} \) such that

\[
\chi^*_1(\mu^0, \chi_1^0, \chi_2^0) \geq f(\mu^0, \chi_1^0, \chi_2^0, \bar{\beta}, \bar{B}) - \epsilon,
\]

and there is \( \delta > 0 \) such that

\[
\bar{\beta} \neq \chi_2 - \bar{B}
\]

263
and

\[ f(\mu, \chi_1, \chi_2, \tilde{\beta}, \tilde{B}) \leq f(\mu^0, \chi^0_1, \chi^0_2, \tilde{\beta}, \tilde{B}) + \epsilon \]

for all \( \mu > 0, \chi_1 > 0, \chi_2 > 0 \) satisfying \( |\mu - \mu^0| < \delta, |\chi_1 - \chi^0_1| < \delta, \) and \( |\chi_2 - \chi^0_2| < \delta \). This implies that

\[ f(\mu, \chi_1, \chi_2, \tilde{\beta}, \tilde{B}) \leq f(\mu^0, \chi^0_1, \chi^0_2, \tilde{\beta}, \tilde{B}) + \epsilon \leq \chi^*_1(\mu^0, \chi^0_1, \chi^0_2) + 2\epsilon, \quad (2.5) \]

and then

\[ \chi^*_1(\mu, \chi_1, \chi_2) \leq \chi^*_1(\mu^0, \chi^0_1, \chi^0_2) + 2\epsilon. \]

Therefore,

\[ \limsup_{(\mu, \chi_1, \chi_2) \to (\mu^0, \chi^0_1, \chi^0_2)} \chi^*(\mu, \chi_1, \chi_2) \leq \chi^*(\mu^0, \chi^0_1, \chi^0_2) \quad \forall \mu^0, \chi^0_1, \chi^0_2 > 0. \]

(2) We first prove (2.2) when \( \chi_1 = \chi_2 \). In this case, we have

\[ f(\mu, \chi_1, \chi_2, \beta, B) = \mu(B + \beta) \left(1 + \frac{(\chi_2 - B - \beta)^2}{4\beta}\right). \]

This implies that

\[ \chi^*_1(\mu, \chi_1, \chi_2) = \inf \left\{ \mu(B + \beta) \left(1 + \frac{(\chi_2 - B - \beta)^2}{4\beta}\right) \mid B > 0, \beta > 0, \beta \neq \chi_2 - B \right\} \]

\[ \leq \inf \left\{ \mu\beta \left(1 + \frac{(\chi_2 - \beta)^2}{4\beta}\right) \mid 0 < \beta < \chi_2 \right\} \]

\[ = \begin{cases} \frac{\mu\chi^2_2}{4} & \text{if } 0 < \chi_2 < 2 \\ \mu(\chi_2 - 1) & \text{if } \chi_2 > 2. \end{cases} \]

Similarly,

\[ \chi^*_2(\mu, \chi_1, \chi_2) \leq \begin{cases} \frac{\mu\chi^2_1}{4} & \text{if } 0 < \chi_1 < 2 \\ \mu(\chi_1 - 1) & \text{if } \chi_1 > 2. \end{cases} \]

Hence (2.2) holds.

Next we prove (2.1) for any \( \mu > 0, \chi_1 > 0 \) and \( \chi_2 > 0 \). Recall that

\[ f(\mu, \chi_1, \chi_2, \beta, B) = \mu(B + \beta) \left(1 + \frac{B(\chi_2 - B - \beta)^2 + (\chi_1 - \chi_2)^2\beta}{4B\beta}\right). \]

Then
\[ f(\mu, \chi_1, \chi_2, \beta, \chi_2) = \mu(\chi_2 + \beta) \left( 1 + \frac{\beta \chi_2 + (x_1 - x_2)^2}{4 \chi_2} \right) \]

and

\[ \chi^*_1(\mu, \chi_1, \chi_2) \leq \inf \left\{ f(\mu, \chi_1, \chi_2, \beta, \chi_2) \mid \beta > 0 \right\} = \frac{\mu (\chi_1 - \chi_2)^2}{4}. \]  \tag{2.6}

Similarly,

\[ \chi^*_2(\mu, \chi_1, \chi_2) \leq \inf \left\{ f(\mu, \chi_2, \chi_1, \beta, \chi_1) \mid \beta > 0 \right\} = \frac{\mu (\chi_2 - \chi_1)^2}{4}. \]  \tag{2.7}

Hence (2.1) holds. \hspace{1cm} \square

Next, we present some properties of the semigroup generated by \(-\Delta + \mu I\) complemented with Neumann boundary condition on \(L^p(\Omega)\). For given \(1 < p < \infty\), let \(X_p = L^p(\Omega)\) and

\[ A_p = -\Delta + \mu I : D(A_p) \subset L^p(\Omega) \rightarrow L^p(\Omega) \]  \tag{2.8}

with

\[ D(A_p) = \left\{ u \in W^{2,p}(\Omega) \mid \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega \right\}. \]

Then \(-A_p\) generates an analytic semigroup on \(L^p(\Omega)\). We denote it by \(e^{-tA_p}\). Note that \(\text{Re} \sigma(A_p) > 0\). Let \(A^\beta_p\) be the fractional power operator of \(A_p\) (see [15, Definition 1.4.1]). Let \(X^\beta_p = D(A^\beta_p)\) with graph norm \(\|u\|_{X^\beta_p} = \|A^\beta_p u\|_{L^p(\Omega)}\) for \(\beta \geq 0\) and \(u \in X^\beta_p\) (see [15, Definition 1.4.7]).

Lemma 2.2.

(i) For each \(p \in (1, \infty)\) and \(\beta \geq 0\), there is \(C_{p,\beta} > 0\) such that

\[ \|A^\beta_p e^{-A_p t}\|_{L^p(\Omega)} \leq C_{p,\beta} t^{-\beta} e^{-\gamma t}, \quad \text{for} \quad t > 0 \]

for some \(\gamma > 0\).

(ii) If \(m \in [0, 1]\) and \(q \in [p, \infty)\) are such that \(m - \frac{N}{q} < 2\beta - \frac{N}{p}\), then \(X^\beta_p \hookrightarrow W^{m,q}(\Omega)\).

(iii) If \(2\beta - \frac{N}{p} > \theta \geq 0\), then \(X^\beta_p \hookrightarrow C^\theta(\Omega)\).

Proof. (i) It follows from [15, Theorem 1.4.3].

(ii) It follows from [15, Theorem 1.6.1].

(iii) It also follows from [15, Theorem 1.6.1]. \hspace{1cm} \square

Lemma 2.3. Let \(\beta \geq 0\), \(p \in (1, \infty)\). Then for any \(\epsilon > 0\) there exists \(C_{p,\beta,\epsilon} > 0\) such that for any \(w \in C^\infty(\Omega)\) we have
\[ \| A_p^{\beta} e^{-t A_p} \nabla \cdot w \|_{L^p(\Omega)} \leq C_{p, \beta, \epsilon} t^{-\beta - \frac{1}{2} - \epsilon} e^{-\gamma t} \| w \|_{L^p(\Omega)} \quad \text{for all } t > 0 \text{ and some } \gamma > 0. \quad (2.9) \]

Consequently, for all \( t > 0 \) the operator \( A_p^{\beta} e^{-t A_p} \nabla \cdot \) admits a unique extension to all of \( L^p(\Omega) \) which is again denoted by \( A_p^{\beta} e^{-t A_p} \nabla \cdot \) and satisfies \((2.9)\) for all \( \mathbb{R}^N \)-valued \( w \in L^p(\Omega) \).

**Proof.** It follows from [19, Lemma 2.1]. \( \square \)

**Remark 2.2.** We remark that the fractional powers of \( A_p \) are used to prove the boundedness of the \( C^0(\bar{\Omega}) \)-norm of the solutions of \((1.2)\) for some \( \theta > 0 \). Such techniques are also used in [19] (see [19, Theorem 4.1]). It should be pointed out that we can prove the boundedness of the \( L^p(\Omega) \)-norm of the solutions of \((1.2)\) for any \( p \geq 1 \) via the standard estimates for the Neumann heat semigroup \( \{e^{tA}\}_{t \geq 0} \) in \( \Omega \) such as

\[
\| e^{tA} u \|_{L^q(\Omega)} \leq C \left( 1 + t^{-\frac{N}{2}} \left( \frac{1}{p} - \frac{1}{q} \right) \right) \| u \|_{L^p(\Omega)},
\]

\[
\| \nabla e^{tA} u \|_{L^q(\Omega)} \leq C \left( 1 + t^{-\frac{1}{2} - \frac{N}{2}} \left( \frac{1}{p} - \frac{1}{q} \right) \right) e^{-\lambda t} \| u \|_{L^p(\Omega)}
\]

for some \( C > 0 \), all \( t > 0 \), \( u \in L^p(\Omega) \), and \( q > p \) (see [36, Lemma 1.3] for some other estimates for the Neumann heat semigroup \( \{e^{tA}\}_{t \geq 0} \)). Due to the presence of the chemotaxis in \((1.2)\), some estimates for the fractional powers of \( A_p \) or some other arguments such as bootstrap arguments are needed to get the \( C^0(\bar{\Omega}) \)-boundedness of the solutions of \((1.2)\) for some \( \theta > 0 \).

We then present some lemmas on the lower and upper bounds of the solutions of

\[
\begin{cases}
-\Delta w + \mu w = \nu u + \lambda v, & x \in \Omega \\
\frac{\partial w}{\partial n} = 0, & x \in \partial \Omega,
\end{cases}
\]

where \( \mu, \nu, \) and \( \lambda \) are positive constants. For given \( u, v \in L^p(\Omega) \), let \( w(\cdot; u, v) \) be the solution of \((2.10)\).

**Lemma 2.4.** Let \( u, v \in C^0(\bar{\Omega}) \) be nonnegative function such that \( \int_{\Omega} u > 0 \) and \( \int_{\Omega} v > 0 \). Then

\[ w(x; u, v) \geq \delta_0 \int_{\Omega} (u + v) > 0 \quad \text{in } \Omega, \]

where \( \delta_0 \) is some positive constant independent of \( u, v \).

**Proof.** It follows from the arguments of [14, Lemma 2.1] and the Gaussian lower bound for the heat kernel of the laplacian with Neumann boundary condition on smooth domain (see [9, Theorem 4]). \( \square \)

**Lemma 2.5.** For any \( p \geq 1 \), there exists \( C_p > 0 \) such that

\[ \max\{\| w(\cdot; u, v) \|_{L^p(\Omega)}, \| \nabla w(\cdot; u, v) \|_{L^p(\Omega)}\} \leq C_p \| u(\cdot) + v(\cdot) \|_{L^p(\Omega)} \quad \forall u, v \in L^p(\Omega). \]
**Proof.** It follows from $L^p$-estimates for elliptic equations (see [2, Theorem 12.1]). □

**Lemma 2.6.** For any nonnegative $u$, $v \in C(\overline{\Omega})$,

\[
\int_{\Omega} \frac{\|\nabla w(\cdot; u, v)\|^2}{w(\cdot; u, v)^2} \leq \mu |\Omega|.
\]

**Proof.** Multiplying (2.10) by $\frac{1}{w}$ and integrating it over $\Omega$ yields that

\[
0 = \int_{\Omega} \frac{1}{w} \cdot \left( \Delta w - \mu w + vu + \lambda v \right) = \int_{\Omega} \frac{\|\nabla w\|^2}{w^2} - \mu |\Omega| + v \int_{\Omega} \frac{u}{w} + \lambda \int_{\Omega} \frac{v}{w}
\]

for all $t \in (0, T_{\text{max}})$. The lemma thus follows. □

Throughout the rest of this section, we assume that $u_0(x)$ and $v_0(x)$ satisfies (1.11) and $(u(t, x), v(t, x), w(t, x)) := (u(t, x; u_0, v_0), v(t, x; u_0, v_0), w(t, x; u_0, v_0))$ is the unique classical solution of (1.2) on the maximal interval $(0, T_{\text{max}}) := (0, T_{\text{max}}(u_0, v_0))$ with the initial condition $(u(0, x), v(0, x)) = (u_0(x), v_0(x))$. Note that

\[
u(t, x), v(t, x), w(t, x) > 0 \quad \forall x \in \Omega, \ t \in (0, T_{\text{max}}).
\]

We now present some upper bound for $\int_{\Omega} u(t, x; u_0, v_0)dx$ and $\int_{\Omega} v(t, x; u_0, v_0)dx$.

**Lemma 2.7.** For any $\tau \in [0, T_{\text{max}})$, the followings hold.

\[
\int_{\Omega} u(t, x; u_0, v_0)dx \leq m^*_1(\tau, u_0, v_0) := \max \left\{ \int_{\Omega} u(\tau, x; u_0, v_0)dx, \frac{a_1 |\Omega|}{b_1} \right\} \quad \forall t \in [\tau, T_{\text{max}}),
\]

and

\[
\int_{\Omega} v(t, x; u_0, v_0)dx \leq m^*_2(\tau, u_0, v_0) := \max \left\{ \int_{\Omega} v(\tau, x; u_0, v_0)dx, \frac{a_2 |\Omega|}{b_2} \right\} \quad \forall t \in [\tau, T_{\text{max}}),
\]

for any $t \in [\tau, T_{\text{max}})$, where $|\Omega|$ is the Lebesgue measure of $\Omega$. Moreover, if $T_{\text{max}}(u_0, v_0) = \infty$, then

\[
\limsup_{t \to \infty} \int_{\Omega} u(t, x; u_0, v_0)dx \leq \frac{a_1 |\Omega|}{b_1} \quad \text{and} \quad \limsup_{t \to \infty} \int_{\Omega} v(t, x; u_0, v_0)dx \leq \frac{a_2 |\Omega|}{b_2}.
\]

**Proof.** By integrating the first equation in (1.2) with respect to $x$, we get that

\[
\frac{d}{dt} \int_{\Omega} u = \int_{\Omega} \Delta u - \chi \int_{\Omega} \nabla \cdot \left( \frac{u}{w} \nabla w \right) + a_1 \int_{\Omega} u - b_1 \int_{\Omega} u^2 - c_1 \int_{\Omega} vu
\]
\[
\leq a_1\int_\Omega u - \frac{b_1}{|\Omega|} \left( \int_\Omega u \right)^2 \quad \forall t \in (0, T_{\text{max}}).
\]

This together with comparison principle for scalar ODEs implies (2.11) and the first inequality in (2.13) when \( T_{\text{max}} = \infty \).

Similarly, we can prove (2.12) and the second inequality in (2.13) when \( T_{\text{max}} = \infty \). \( \square \)

3. Proofs of Theorems 1.1-1.3

In this section, we investigate local lower bound of the combined mass and local \( L^p \)- and \( C^0 \)-boundedness of classical solutions of (1.2); the global existence of classical solutions of (1.2); and prove Theorems 1.1-1.3.

Throughout this section, for given \( u_0, v_0 \) satisfying (1.11), we put

\[
(u(t, x), v(t, x), w(t, x)) := (u(t, x; u_0, v_0), v(t, x; u_0, v_0), w(t, x; u_0, v_0)),
\]

and if no confusion occurs, we may drop \((t, x)\) in \(u(t, x)\) (resp. \(v(t, x), w(t, x)\)).

3.1. Lower bound of the combined mass on bounded intervals and proof of Theorem 1.1

In this subsection, we study the lower bound of the combined mass on bounded intervals and proof of Theorem 1.1.

We first present a lemma on \( \int_\Omega \ln(u + v)dx \). Note that, by (1.6), \( u(t, x; u_0, v_0) + v(t, x; u_0, v_0) > 0 \) for all \( t \in (0, T_{\text{max}}(u_0, v_0)) \) and \( x \in \Omega \). Hence \( \int_\Omega \ln(u(t, x; u_0, v_0) + v(t, x; u_0, v_0))dx \) is well defined for \( t \in (0, T_{\text{max}}(u_0, v_0)) \).

**Lemma 3.1.** For any \( u_0(x) \) and \( v_0(x) \) satisfying (1.11), there exists \( K = K(u_0, v_0) > 0 \) such that

\[
\frac{d}{dt} \int_\Omega \ln(u(t, x; u_0, v_0) + v(t, x; u_0, v_0))dx \geq -K \quad \text{for all } t \in (0, T_{\text{max}}(u_0, v_0)).
\]

**Proof.** First, multiplying (2.3) by \( \frac{1}{u+v} \) and then integrating on \( \Omega \) yields that

\[
\frac{d}{dt} \int_\Omega \ln(u + v) \geq \int_\Omega \frac{|\nabla (u + v)|^2}{(u + v)^2} - \int_\Omega \frac{\chi_1 u + \chi_2 v}{(u + v)^2} \cdot \nabla \frac{\nabla w}{w} \bigg|_t \bigg. \\
+ \int_\Omega \frac{a_1 u + a_2 v}{u + v} - \int_\Omega \frac{b_1 u^2 + b_2 v^2}{u + v} - \int_\Omega \frac{(c_1 + c_2)uv}{u + v}
\]

for all \( t \in (0, T_{\text{max}}(u_0, v_0)) \). By Young’s inequality and Lemma 2.6, we have

\[
I = \int_\Omega \frac{\chi_1 u + \chi_2 v}{(u + v)^2} \cdot \nabla \frac{\nabla w}{w} \leq \int_\Omega \frac{|\nabla (u + v)|^2}{(u + v)^2} + \max\{\chi_1^2, \chi_2^2\} 4 \int_\Omega \frac{|\nabla w|^2}{w^2}
\]
\leq \int_{\Omega} \frac{\left| \nabla (u + v) \right|^2}{(u + v)^2} + \frac{\mu |\Omega| \max\{\chi_1^2, \chi_2^2\}}{4}.

Then by Lemma 2.7 we have

\frac{d}{dt} \int_{\Omega} \ln (u + v) \geq -\frac{\mu |\Omega| \max\{\chi_1^2, \chi_2^2\}}{4} + a_{\min |\Omega|} - (b_{\max} + c_{\max}) (m_1^*(0, u_0, v_0)

+ m_2^*(0, u_0, v_0))

for all \( t \in (0, T_{\max}) \). The lemma is thus proved. \( \Box \)

We now prove Theorem 1.1.

**Proof of Theorem 1.1.** Fix a \( \tau \in (0, T_{\max}(u_0, v_0)) \). It is clear that

\[ \inf_{0 \leq t \leq \tau} \int_{\Omega} (u(t, x; u_0, v_0) + v(t, x; u_0, v_0)) \, dx > 0. \]

It then suffices to prove that there exist \( C = C(T) > 0 \) such that

\[ \int_{\Omega} (u(t, x; u_0, v_0) + v(t, x; u_0, v_0)) \, dx \geq C(T) \quad \text{for all} \ t \in (\tau, \hat{T}), \quad (3.1) \]

where \( \hat{T} = \min\{T, T_{\max}(u_0, v_0)\} \).

Note that \( L := \int_{\Omega} \ln (u(\tau, x; u_0, v_0) + v(\tau, x; u_0, v_0)) \, dx \) is finite. By Lemma 3.1, there exists \( K = K(u_0, v_0) > 0 \) such that

\[ \frac{d}{dt} \int_{\Omega} \ln (u(t, x; u_0, v_0) + v(t, x; u_0, v_0)) \, dx \geq -K \]

for all \( t \in (0, T_{\max}(u_0, v_0)) \). We thus have that

\[ \int_{\Omega} \ln (u(t, x; u_0, v_0) + v(t, x; u_0, v_0)) \, dx \]

\[ \geq \int_{\Omega} \ln (u(\tau, x; u_0, v_0) + v(\tau, x; u_0, v_0)) \, dx - K \cdot (t - \tau) \]

\[ \geq L - K \cdot (\hat{T} - \tau) = C(K, L, \tau) \quad \text{for all} \ t \in (\tau, \hat{T}). \]

Therefore Jensen’s inequality asserts that
\[
\int_{\Omega} \ln \left( u(t, x; u_0, v_0) + v(t, x; u_0, v_0) \right) dx = |\Omega| \cdot \frac{d}{dt} \int_{\Omega} \ln \left( u(t, x; u_0, v_0) + v(t, x; u_0, v_0) \right) dx \\
\leq |\Omega| \cdot \ln \left( \int_{\Omega} \ln \left( u(t, x; u_0, v_0) + v(t, x; u_0, v_0) \right) dx \right)
\]

for all \( t \in (0, T_{\text{max}}(u_0, v_0)) \), which implies

\[
\int_{\Omega} (u(t, x; u_0, v_0) + v(t, x; u_0, v_0)) dx \\
\geq |\Omega| \cdot \exp \left( \frac{1}{|\Omega|} \int_{\Omega} \ln \left( u(t, x; u_0, v_0) + v(t, x; u_0, v_0) \right) dx \right) \\
\geq |\Omega| \cdot e^{1/|\Omega|} \cdot C(K, L, \tau) \quad \text{for all} \quad t \in (\tau, \hat{T}),
\]

which implies (3.1). The theorem is thus proved. \( \square \)

3.2. Local \( L^p \)- and \( C^\theta \)-boundedness and proof of Theorem 1.2

In this subsection, we study the \( L^p \)- and \( C^\theta \)-boundedness of classical solutions of (1.2) and prove Theorem 1.2. We first make some observations.

Fix \( p > 2 \). First, observe that

\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} (u + v)^p = - (p - 1) \int_{\Omega} (u + v)^{p-2} |\nabla (u + v)|^2 \\
+ (p - 1) \int_{\Omega} (u + v)^{p-2} \frac{\chi_1 u + \chi_2 v}{w} \nabla (u + v) \cdot \nabla w \\
+ \int_{\Omega} (u + v)^{p-1} (a_1 u + a_2 v) - \int_{\Omega} (u + v)^{p-1} (b_1 u^2 + b_2 v^2) \\
- \int_{\Omega} (u + v)^{p-1} (c_1 + c_2) uv \\
\leq - (p - 1) \int_{\Omega} (u + v)^{p-2} |\nabla (u + v)|^2 \\
+ (p - 1) \int_{\Omega} (u + v)^{p-2} \frac{\chi_1 u + \chi_2 v}{w} \nabla (u + v) \cdot \nabla w \\
+ a_{\text{max}} \int_{\Omega} (u + v)^p - \min\{b_{\text{min}}, c_{\text{min}}\} \int_{\Omega} (u + v)^{p+1}
\]

(3.2)
for \( t \in (0, T_{\text{max}}(u_0, v_0)) \). To prove Theorem 1.2, it is then essential to provide proper estimates for the integral \( \int_{\Omega} (u + v)^{p-2} \frac{\partial (u + v)}{w} \cdot \nabla (u + v) \cdot \nabla w \).

Next, observe that

\[
\int_{\Omega} \frac{(u + v)^{p-2}(\chi_1 u + \chi_2 v)}{w} \nabla (u + v) \cdot \nabla w = \chi_2 \int_{\Omega} \frac{(u + v)^{p-1}}{w} \nabla (u + v) \cdot \nabla w + (\chi_1 - \chi_2) \int_{\Omega} \frac{(u + v)^{p-2}}{w} u \nabla (u + v) \cdot \nabla w.
\]

Since \( u^2 \leq (u + v)^2 \), by Young’s inequality, for any \( \varepsilon > 0 \), we have that

\[
| (\chi_1 - \chi_2) \int_{\Omega} \frac{(u + v)^{p-2}}{w} u \nabla (u + v) \cdot \nabla w | \leq \varepsilon \int_{\Omega} (u + v)^{p-2} | \nabla (u + v) |^2 + \frac{(\chi_1 - \chi_2)^2}{4\varepsilon} \int_{\Omega} (u + v)^{p-2} u^2 \frac{| \nabla w |^2}{w^2} \leq \varepsilon \int_{\Omega} (u + v)^{p-2} | \nabla (u + v) |^2 + \frac{(\chi_1 - \chi_2)^2}{4\varepsilon} \int_{\Omega} (u + v)^{p} \frac{| \nabla w |^2}{w^2}.
\]

Multiplying the third equation in (1.2) by \( \frac{(u + v)^p}{w} \) and then integrating over \( \Omega \) yields that

\[
0 = \int_{\Omega} \frac{(u + v)^p}{w} \cdot (\Delta w - \mu w + \nu u + \lambda v)
= -\int_{\Omega} \frac{p(u + v)^{p-1} w \nabla (u + v) - (u + v)^p \nabla w}{w^2} \cdot \nabla w
- \mu \int_{\Omega} (u + v)^p + \nu \int_{\Omega} \frac{u(u + v)^p}{w} + \lambda \int_{\Omega} \frac{v(u + v)^p}{w}
\]

for all \( t \in (0, T_{\text{max}}(u_0, v_0)) \). Thus we have,

\[
\int_{\Omega} (u + v)^p \frac{| \nabla w |^2}{w^2} \leq p \int_{\Omega} \frac{(u + v)^{p-1}}{w} \nabla (u + v) \cdot \nabla w + \mu \int_{\Omega} (u + v)^p
\]

for all \( t \in (0, T_{\text{max}}(u_0, v_0)) \). Therefore, for any \( \varepsilon > 0 \),

\[
\int_{\Omega} \frac{(u + v)^{p-2}(\chi_1 u + \chi_2 v)}{w} \nabla (u + v) \cdot \nabla w
\]
\[
\leq \varepsilon \int_{\Omega} (u + v)^{p-2} |\nabla (u + v)|^2 dx \\
+ \left( \chi_2 + \frac{p(\chi_1 - \chi_2)^2}{4\varepsilon} \right) \int_{\Omega} \frac{(u + v)^{p-1}}{w} \nabla (u + v) \cdot \nabla w \\
+ \frac{(\chi_1 - \chi_2)^2}{4\varepsilon} \mu \int_{\Omega} (u + v)^p.
\]

(3.3)

To provide proper estimates for \( \int_{\Omega} (u + v)^{p-2} \frac{\chi_1 w + \chi_2 v}{w} \nabla (u + v) \cdot \nabla w \), it is then essential to provide proper estimates for \( \int_{\Omega} \frac{(u + v)^{p-1}}{w} \nabla (u + v) \cdot \nabla w \). We have the following two propositions on the estimates for \( \int_{\Omega} \frac{(u + v)^{p-1}}{w} \nabla (u + v) \cdot \nabla w \) for some \( p > 3N \).

**Proposition 3.1.** There is \( p > 3N \) such that for every given \( \varepsilon > 0 \) and \( 0 < \varepsilon_0 \ll 1 \), there is \( C > 0 \) such that

\[
\int_{\Omega} \frac{(u + v)^{p-1}}{w} \nabla (u + v) \cdot \nabla w \leq \varepsilon \int_{\Omega} (u + v)^{p-2} |\nabla (u + v)|^2 \\
+ C \left( \int_{\Omega} (u + v)^{p+1-\varepsilon_0} + \int_{\Omega} \left( \frac{|\nabla w|}{w} \right)^{2p+2-\varepsilon_0} \right) + C
\]

(3.4)

for any \( u_0, v_0 \in C^0(\overline{\Omega}) \) satisfying (1.11), and \( t \in (0, T_{\text{max}}(u_0, v_0)) \).

**Remark 3.1.** We remark that \( p > 3N \) in the above proposition can be any number satisfying (5.2)-(5.7) in the appendix. Given any such \( p, \varepsilon_0 > 0 \) in the above proposition can be any positive number satisfying \( \varepsilon_0 < 2 \), and (5.8), (5.9) in the appendix.

**Proposition 3.2.** Let \( p \geq 3 \) and \( p - \sqrt{2p-3} < k < p + \sqrt{2p-3} \). There are \( M, \tilde{M} > 0 \) such that

\[
\int_{\Omega} \frac{|\nabla w|^2}{w^k} \leq M \int_{\Omega} \frac{(\nu u + \lambda v)^p}{w^{k-p}} + \tilde{M} \int_{\Omega} w^{2p-k} \quad \forall \ t \in (0, T_{\text{max}}(u_0, v_0)).
\]

**Remark 3.2.** We remark that the estimate in Proposition 3.2 plays a crucial role in the proof of the boundedness of the solutions of (1.2). The proof of Proposition 3.2 strongly relies on the fact that the third equation in (1.2) is elliptic. It therefore requires new techniques to investigate the global existence and boundedness of solutions of (1.31).

In the rest of this subsection, we prove Theorem 1.2 by applying Propositions 3.1 and 3.2. Proposition 3.1 can be proved by the similar arguments as those in [26, Proposition 4.1], and Proposition 3.2 can be proved by the similar arguments as those in [26, Proposition 4.2]. For the reader’s convenience, we will provide the outline of the proofs of Propositions 3.1 and 3.2 in the appendix.
Proof of Theorem 1.2. (1) First, let \( p \) be as in Proposition 3.1. By (3.2) and (3.3), for any \( \varepsilon > 0 \), we have
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} (u + v)^p \leq -(p - 1) \int_{\Omega} (u + v)^{p-2} |\nabla (u + v)|^2 + (p - 1)\varepsilon \int_{\Omega} (u + v)^{p-2} |\nabla (u + v)|^2 dx
\]
\[
+ (p - 1) \left( \chi_2 + \frac{p(\chi_1 - \chi_2)^2}{4\varepsilon} \right) \int_{\Omega} \frac{(u + v)^{p-1}}{w} \nabla (u + v) \cdot \nabla w
\]
\[
+ (p - 1) \left( \frac{\chi_1 - \chi_2)^2}{4\varepsilon} \mu \int_{\Omega} (u + v)^p
\]
\[
+ a_{\max} \int_{\Omega} (u + v)^p - \min\{b_{\min}, c_{\min}\} \int_{\Omega} (u + v)^{p+1}
\]
for all \( t \in (0, T_{\max}(u_0, v_0)) \).

Next, let \( \hat{T} = \min\{T, T_{\max}(u_0, v_0)\} \) and \( \tau \in (0, T_{\max}(u_0, v_0)) \). By (3.5) and Proposition 3.1, there are \( 0 < \varepsilon_0 \ll 1 \) and \( C_1 > 0 \) such that
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} (u + v)^p \leq C_1 \left( \int_{\Omega} (u + v)^{p+1-\varepsilon_0} + \int_{\Omega} \left( \frac{\nabla w}{w} \right)^{2p+2-\varepsilon_0} \right) + C_1
\]
\[
+ (p - 1) \left( \frac{\chi_1 - \chi_2)^2}{4\varepsilon} \mu \int_{\Omega} (u + v)^p
\]
\[
+ a_{\max} \int_{\Omega} (u + v)^p - \min\{b_{\min}, c_{\min}\} \int_{\Omega} (u + v)^{p+1}
\]
\[
\leq \inf_{\tau \leq t < \hat{T}, x \in \Omega} \frac{C_1}{w^{p+1-\varepsilon_0/2}(t, x)} \int_{\Omega} \frac{|\nabla w|^{2p+2-\varepsilon_0}}{w^{p+1-\varepsilon_0/2}}
\]
\[
+ C_1 \int_{\Omega} (u + v)^{p+1-\varepsilon_0} + C_1 + (p - 1) \left( \frac{\chi_1 - \chi_2)^2}{4\varepsilon} \mu \int_{\Omega} (u + v)^p
\]
\[
+ a_{\max} \int_{\Omega} (u + v)^p - \min\{b_{\min}, c_{\min}\} \int_{\Omega} (u + v)^{p+1}
\]
for all \( t \in [\tau, \hat{T}) \). This together with Proposition 3.2 implies that there is \( C_2 > 0 \) such that
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} (u + v)^p \leq \inf_{\tau \leq t < \hat{T}, x \in \Omega} \frac{C_2}{w^{p+1-\varepsilon_0/2}(t, x)} \left( \int_{\Omega} (u + v)^{p+1-\varepsilon_0/2} + \int_{\Omega} w^{p+1-\varepsilon_0/2} \right)
\]
\[
+ C_1 \int_{\Omega} (u + v)^{p+1-\varepsilon_0} + C_1 + (p - 1) \left( \frac{\chi_1 - \chi_2)^2}{4\varepsilon} \mu \int_{\Omega} (u + v)^p
\]
\[
+ a_{\text{max}} \int_{\Omega} (u + v)^p - \min\{b_{\text{min}}, c_{\text{min}}\} \int_{\Omega} (u + v)^{p+1}
\] (3.7)

for all \( t \in [\tau, \hat{T}) \).

Now, by Lemma 2.5 and Young’s inequality, there is \( C_3 \geq \max\{C_1, C_2\} \) such that

\[
\int_{\Omega} w^{p + 1 - \varepsilon_0 / 2} \leq \frac{C_3}{C_2} \int_{\Omega} (u + v)^{p + 1 - \varepsilon_0 / 2} \quad \text{and} \quad \int_{\Omega} (u + v)^{p + 1 - \varepsilon_0 / 2} \leq \frac{C_3}{C_1} \int_{\Omega} (u + v)^{p + 1 - \varepsilon_0 / 2} + C_3.
\]

This together with (3.7) implies that

\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} (u + v)^p \leq \left( \frac{2C_3}{\inf_{\tau \leq t < \hat{T}, x \in \Omega} w^{p + 1 - \varepsilon_0 / 2}(t, x)} + C_3 \right) \int_{\Omega} (u + v)^{p + 1 - \varepsilon_0 / 2} + C_1 + C_3 + (p - 1) \frac{(\chi_1 - \chi_2)^2}{4\varepsilon} \mu \int_{\Omega} (u + v)^p
\]

\[
+ a_{\text{max}} \int_{\Omega} (u + v)^p - \min\{b_{\text{min}}, c_{\text{min}}\} \int_{\Omega} (u + v)^{p+1}
\] (3.8)

for \( t \in [\tau, \hat{T}) \). This together with Young’s inequality implies that there is a positive constant \( C_4 > 0 \) such that

\[
\frac{d}{dt} \int_{\Omega} (u + v)^p(t, x)dx \leq - \int_{\Omega} (u + v)^p(t, x)dx + C_4 \left( \frac{1}{\inf_{\tau \leq t < \hat{T}, x \in \Omega} w^{p + 1 - \varepsilon_0 / 2}(t, x)} + 1 \right)^{2(p+1)/\varepsilon_0} + C_4
\] (3.9)

for all \( t \in [\tau, \hat{T}) \).

Finally, by (3.9) and the comparison principle for scalar ODEs, we have

\[
\int_{\Omega} (u + v)^p(t, x)dx \leq e^{-(t-\tau)} \int_{\Omega} (u + v)^p(\tau, x)dx + C_4 \left( \frac{1}{\inf_{\tau \leq t < \hat{T}, x \in \Omega} w^{p + 1 - \varepsilon_0 / 2}(t, x)} + 1 \right)^{2(p+1)/\varepsilon_0} + C_4
\]

and

274
\[
\int_{\Omega} (u + v)^p(t, x) \, dx \leq \max \left\{ e^{-(t-\tau)} \int_{\Omega} (u + v)^p(\tau, x) \, dx, C_4 \left( \inf_{\tau \leq t < \hat{T}, x \in \Omega} \frac{1}{w^{p+1-\epsilon_0/2}(t, x)} + 1 \right)^{2(p+1)/\epsilon_0} + C_4 \right\}
\]
for any \( t \in [\tau, \hat{T}) \). Theorem 1.2(1) thus follows with \( M_1 := C_4 \).

(2) For given \( p > 2N \) and \( 0 < \theta < 1 - \frac{2N}{p} \), choose \( \beta \in (0, \frac{1}{2}) \) such that \( 2\beta - \frac{2N}{p} > \theta \). By Lemma 2.2, there is \( C_{p,\beta} > 0 \) such that for any \( u_0, v_0 \) satisfying (1.11),
\[
\|u(\cdot, \cdot) + v(\cdot, \cdot)\|_{\mathcal{C}^0(\Omega)} \leq C_{p,\beta} \|u(t, \cdot) + v(t, \cdot)\|_{X_{p,\beta}} \quad \forall 0 < t < T_{\text{max}}.
\]
(3.10)
Note that in view of (2.3), we have that
\[
\|u(t, \cdot) + v(t, \cdot)\|_{X_{p,\beta}} \leq \|A_p^\frac{\beta}{2} e^{-A_p^\frac{\beta}{2} (t-\tau)} (u(\tau, \cdot; u_0, v_0) + v(\tau, \cdot; u_0, v_0))\|_{L^{p}/2(\Omega)}
\]
\[
+ \int_\tau^t \|A_p^\frac{\beta}{2} e^{-A_p^\frac{\beta}{2} (t-s)} \nabla \cdot \left( \frac{\chi_1 u(s, \cdot) + \chi_2 v(s, \cdot)}{w(s, \cdot)} \nabla w(s, \cdot) \right)\|_{L^{p}/2(\Omega)} \, ds
\]
\[
+ \int_\tau^t \|A_p^\frac{\beta}{2} e^{-A_p^\frac{\beta}{2} (t-s)} g(u(s, \cdot), v(s, \cdot))\|_{L^{p}/2(\Omega)} \, ds
\]
(3.11)
for any \( 0 \leq \tau < T_{\text{max}}(u_0, v_0) \) and \( \tau < t < T_{\text{max}}(u_0, v_0) \), where \( A_p^\frac{\beta}{2} = -\Delta + \mu I \) is defined as in (2.8) with \( p \) being replaced by \( \frac{p}{\beta} \), and
\[
g(u, v) = (\mu + a_1)u + (\mu + a_2)v - (b_1 u^2 + b_2 v^2) - (c_1 + c_2)uv.
\]
By Lemma 2.2 again, there is \( \gamma > 0 \) such that
\[
\|A_p^\frac{\beta}{2} e^{-A_p^\frac{\beta}{2} (t-\tau)} (u(\tau, \cdot; u_0, v_0) + v(\tau, \cdot; u_0, v_0))\|_{L^{p}/2(\Omega)}
\]
\[
\leq C_{p,\beta} (t-\tau)^{-\beta} e^{-\gamma(t-\tau)} \|u(\tau, \cdot; u_0, v_0) + v(\tau, \cdot; u_0, v_0)\|_{L^{p}/2} \quad \forall \tau < t < T_{\text{max}}(u_0, v_0).
\]
(3.12)
By Lemma 2.3 and Lemma 2.5, for any \( \epsilon > 0 \), there are \( \gamma > 0, C_p > 0 \) and \( C_{p,\beta,\epsilon} > 0 \) such that
\[
\int_\tau^t \|A_p^\frac{\beta}{2} e^{-A_p^\frac{\beta}{2} (t-s)} \nabla \cdot \left( \frac{\chi_1 u(s, \cdot) + \chi_2 v(s, \cdot)}{w(s, \cdot)} \nabla w(s, \cdot) \right)\|_{L^{p}/2(\Omega)} \, ds
\]
\[
\leq C_{p,\beta,\epsilon} \int_\tau^t (1 + (t-s)^{-\beta -\frac{\gamma}{2} -\epsilon}) e^{-\gamma(t-s)} \left\| \frac{\chi_1 u(s, \cdot) + \chi_2 v(s, \cdot)}{w(s, \cdot)} \nabla w(s, \cdot) \right\|_{L^{p}/2(\Omega)} \, ds
\]
275
\[
\begin{align*}
&\leq C_{p,\beta,\epsilon} (\chi_1 + \chi_2) \int_{\tau}^{t} \left( 1 + (t-s)^{-\frac{\beta}{2} - \frac{1}{2} - \epsilon} e^{-\gamma(t-s)} \right) \| u(s, \cdot) + v(s, \cdot) \|_{L^p(\Omega)} \| \nabla w(s, \cdot) \|_{L^p(\Omega)} \inf_{\tau \leq t < \min \{ T, T_{\text{max}} \}, x \in \Omega} w(t, x) \, ds \\
&\leq C_p C_{p,\beta,\epsilon} (\chi_1 + \chi_2) \int_{\tau}^{t} \left( 1 + (t-s)^{-\frac{\beta}{2} - \frac{1}{2} - \epsilon} e^{-\gamma(t-s)} \right) \| u(s, \cdot) + v(s, \cdot) \|_{L^p(\Omega)}^2 \inf_{\tau \leq t < \min \{ T, T_{\text{max}} \}, x \in \Omega} w(t, x) \, ds.
\end{align*}
\] (3.13)

By Lemma 2.2 again,

\[
\int_{\tau}^{t} \| A_{\frac{x}{2}} e^{-A\frac{x}{2}(t-s)} g(u(s, \cdot), v(s, \cdot)) \|_{L^{p/2}(\Omega)} \, ds \\
\leq C_{p,\beta} \int_{\tau}^{t} (t-s)^{-\frac{\beta}{2} - \gamma(t-s)} \left\{ \mu |\Omega|^{2/p} + a_{\text{max}} \| u(s, \cdot) + v(s, \cdot) \|_{L^{p/2}} \\
+ b_{\text{max}} \| (u(s, \cdot) + v(s, \cdot))^2 \|_{L^{p/2}} + \frac{c_{\text{max}}}{2} \| (u(s, \cdot) + v(s, \cdot))^2 \|_{L^{0/2}} \right\} \, ds.
\] (3.14)

(1.21) then follows from (3.10) to (3.14). \(\square\)

3.3. Global existence and proof of Theorem 1.3

In this subsection, we study the global existence of classical solutions of (1.2) and prove Theorem 1.3.

Proof of Theorem 1.3. We prove Theorem 1.3 by contradiction. Assume that \(T_{\text{max}} < \infty\). Then by Lemma 2.4 and Theorem 1.1, there is \(\delta > 0\) such that

\[
w(t, x) \geq \delta \quad \text{for all } x \in \Omega \quad \text{and } t \in (0, T_{\text{max}}),
\] (3.15)

and then by Proposition 1.1, we have

\[
\limsup_{t \nearrow T_{\text{max}}} \| u(t, \cdot) + v(t, \cdot) \|_{C^0(\bar{\Omega})} = \infty.
\]

But, by (3.15) and Theorem 1.2, we have

\[
\lim_{t \nearrow T_{\text{max}}} \| u(t, \cdot) + v(t, \cdot) \|_{C^0(\bar{\Omega})} < \infty,
\]

which is a contradiction. Therefore \(T_{\text{max}} = \infty\). \(\square\)
4. Proofs of Theorems 1.4 and 1.5

In this section, we study the boundedness of $\int_{\Omega} (u + v)^{-q}$ and $\int_{\Omega} (u + v)^p$ and prove Theorems 1.4 and 1.5.

Throughout this section, for given $u_0, v_0$ satisfying (1.11), we put

$$(u(t, x), v(t, x), w(t, x)) := (u(t, x; u_0, v_0), v(t, x; u_0, v_0), w(t, x; u_0, v_0)),$$

and if no confusion occurs, we may drop $(t, x)$ in $u(t, x)$ (resp. $v(t, x), w(t, x)$).

4.1. Proof of Theorem 1.4

In this subsection, we study the boundedness of $\int_{\Omega} (u + v)^{-q}$ and $\int_{\Omega} (u + v)^p$ and prove Theorem 1.4.

We first prove a lemma.

Lemma 4.1. Let $\mu > 0, \chi_1 > 0$ and $\chi_2 > 0$ be given.

(1) For any $B > 0$ and $0 < \beta \neq \chi_2 - B$,

$$\frac{1}{q} \frac{d}{dt} \int_{\Omega} (u + v)^{-q} \leq \left( f(\mu, \chi_1, \chi_2, \beta, B) - a_{\min} \right) \int_{\Omega} (u + v)^{-q}$$

$$+ (b_{\max} + c_{\max}) \int_{\Omega} (u + v)^{-q + 1}, \tag{4.1}$$

where $f(\mu, \chi_1, \chi_2, \beta, B)$ is as in (1.13) and

$$q = \frac{4B\beta}{B(\chi_2 - B - \beta)^2 + (\chi_1 - \chi_2)^2 \beta}. \tag{4.2}$$

(2) For any $B > 0$ and $0 < \beta \neq \chi_1 - B$,

$$\frac{1}{q} \frac{d}{dt} \int_{\Omega} (u + v)^{-q} \leq \left( f(\mu, \chi_2, \chi_1, \beta, B) - a_{\min} \right) \int_{\Omega} (u + v)^{-q}$$

$$+ (b_{\max} + c_{\max}) \int_{\Omega} (u + v)^{-q + 1}, \tag{4.3}$$

where

$$q = \frac{4B\beta}{B(\chi_1 - B - \beta)^2 + (\chi_1 - \chi_2)^2 \beta}. \tag{4.4}$$

Remark 4.1. Like Proposition 3.2, Lemma 4.1 also plays a crucial role in the proof of the boundedness of the solutions of (1.2). Again, the proof of Lemma 4.1 strongly relies on the fact that the third equation in (1.2) is elliptic. Therefore new techniques need to be developed to investigate the global existence and boundedness of solutions of (1.31).
Proof of Lemma 4.1. (1) First, by (2.4), we have

\[
\frac{1}{q} \frac{d}{dt} \int_{\Omega} (u + v)^{-q} \leq - (q + 1) \int_{\Omega} (u + v)^{-q-2} |\nabla (u + v)|^2 \\
+ (q + 1) \chi_2 \int_{\Omega} \frac{(u + v)^{-q-1}}{w} \nabla (u + v) \cdot \nabla w \\
+ \frac{(\chi_1 - \chi_2)^2 q(q + 1)}{4B} \int_{\Omega} (u + v)^{-q-2} |\nabla (u + v)|^2 \\
+ \frac{B(q + 1)}{q} \int_{\Omega} (u + v)^{-q} \frac{|\nabla w|^2}{w^2} \\
- a_{\min} \int_{\Omega} (u + v)^{-q} + (b_{\max} + c_{\max}) \int_{\Omega} (u + v)^{-q+1}
\]

for any \( q > 0 \) and \( B > 0 \).

Next, multiplying the third equation in (1.2) by \( \frac{(u + v)^{-q}}{w} \) and then integrating over \( \Omega \) with respect to \( x \), we obtain that

\[
0 = \int_{\Omega} \frac{(u + v)^{-q}}{w} \cdot (\Delta w - \mu w + vu + \lambda v) \\
= - \int_{\Omega} \frac{(-q)(u + v)^{-q-1} w \nabla (u + v) - (u + v)^{-q} \nabla w}{w^2} \cdot \nabla w \\
- \mu \int_{\Omega} (u + v)^{-q} + v \int_{\Omega} \frac{u(u + v)^{-q}}{w} + \lambda \int_{\Omega} \frac{v(u + v)^{-q}}{w}
\]

for all \( t \in (0, \infty) \). Thus we have,

\[
q \int_{\Omega} \frac{(u + v)^{-q-1}}{w} \nabla (u + v) \cdot \nabla w + \int_{\Omega} (u + v)^{-q} \frac{|\nabla w|^2}{w^2} \leq \mu \int_{\Omega} (u + v)^{-q}
\]

for all \( t \in (0, \infty) \). It then follows that

\[
\frac{B}{q} \int_{\Omega} (u + v)^{-q} \frac{|\nabla w|^2}{w^2} \leq \frac{B \mu}{q} \int_{\Omega} (u + v)^{-q} - B \int_{\Omega} \frac{(u + v)^{-q-1}}{w} \nabla (u + v) \cdot \nabla w.
\]

This together with (4.5) implies that

[278]
\[
\frac{1}{q} \frac{d}{dt} \int_{\Omega} (u + v)^{-q} \leq -(q + 1) \int_{\Omega} (u + v)^{-q-2} |\nabla (u + v)|^2
\]
\[
+ (q + 1)(\chi_2 - B) \int_{\Omega} \frac{(u + v)^{-q-1}}{w} \nabla (u + v) \cdot \nabla w
\]
\[
+ \frac{\chi_1 - \chi_2}{4B} (q + 1) \int_{\Omega} (u + v)^{-q-2} |\nabla (u + v)|^2
\]
\[
+ \frac{B(q + 1)\mu}{q} \int_{\Omega} (u + v)^{-q}
\]
\[
- a_{\min} \int_{\Omega} (u + v)^{-q} + (b_{\max} + c_{\max}) \int_{\Omega} (u + v)^{-q+1}
\]
(4.7)

for any \( q > 0 \) and \( B > 0 \).

Now, let \( \tilde{\chi} = \chi_2 - B \). Note that, when \( B > 0 \) and \( q > 0 \) are such that \( D := \frac{\chi_1 - \chi_2}{4B} - q < 1 \), for any \( \beta > 0 \), by Young’s inequality and (4.6), we have

\[
\tilde{\chi} \int_{\Omega} \frac{(u + v)^{-q-1}}{w} \nabla (u + v) \cdot \nabla w = (\tilde{\chi} - \beta) \int_{\Omega} \frac{(u + v)^{-q-1}}{w} \nabla (u + v) \cdot \nabla w
\]
\[
+ \beta \int_{\Omega} \frac{(u + v)^{-q-1}}{w} \nabla (u + v) \cdot \nabla w
\]
\[
\leq (1 - D) \int_{\Omega} (u + v)^{-q-2} |\nabla (u + v)|^2
\]
\[
+ \frac{|\tilde{\chi} - \beta|^2}{4(1 - D)} \int_{\Omega} (u + v)^{-q} \frac{|\nabla w|^2}{w^2}
\]
\[
+ \frac{\beta \mu}{q} \int_{\Omega} (u + v)^{-q} - \frac{\beta}{q} \int_{\Omega} (u + v)^{-q} \frac{|\nabla w|^2}{w^2}.
\]

This together with (4.7) implies that

\[
\frac{1}{q} \frac{d}{dt} \int_{\Omega} (u + v)^{-q} \leq (q + 1) \left( \frac{B|\tilde{\chi} - \beta|^2}{4B - (\chi_1 - \chi_2)^2} - \frac{\beta}{q} \right) \int_{\Omega} (u + v)^{-q} \frac{|\nabla w|^2}{w^2}
\]
\[
+ (q + 1) \left( \frac{B\mu}{q} + \frac{\beta \mu}{q} \right) \int_{\Omega} (u + v)^{-q}
\]
\[
- a_{\min} \int_{\Omega} (u + v)^{-q} + (b_{\max} + c_{\max}) \int_{\Omega} (u + v)^{-q+1}
\]
(4.8)
for any \( \beta > 0 \) and \( B > 0, q > 0 \) with \( \frac{(\chi_1 - \chi_2)^2}{4B} q < 1 \).

Finally, for any \( B > 0 \) and \( \beta > 0, \beta \neq \tilde{\chi} \), let \( q \) be as in (4.2). Then

\[
\frac{(\chi_1 - \chi_2)^2}{4B} q < 1,
\]

and

\[
\frac{B|\tilde{\chi} - \beta|^2}{4B - (\chi_1 - \chi_2)^2 q} - \frac{\beta}{q} = 0.
\]

Then by (4.8), we have

\[
\frac{1}{q} \frac{d}{dt} \int_{\Omega} (u + v)^{-q} \leq \left( (B\mu + \beta\mu) \left( 1 + \frac{1}{q} \right) - a_{\min} \right) \int_{\Omega} (u + v)^{-q} + (b_{\max} + c_{\max}) \int_{\Omega} (u + v)^{-q+1}.
\]

Note that

\[
(B\mu + \beta\mu) \left( 1 + \frac{1}{q} \right) = f(\mu, \chi_1, \chi_2, \beta, B).
\]

This implies that (4.1) holds.

(2) It can be proved by the similar arguments as in (1). \( \square \)

We now prove Theorem 1.4.

**Proof of Theorem 1.4.** (1) First, without loss of generality, we assume that

\[
\chi^*(\mu, \chi_1, \chi_2) = \min\{\chi^*_1(\mu, \chi_1, \chi_2), \chi^*_2(\mu, \chi_1, \chi_2)\} = \chi^*_1(\mu, \chi_1, \chi_2).
\]

Let \( \epsilon_0 = a_{\min} - \chi^*(\mu, \chi_1, \chi_2) \). Then by the definition of \( \chi^*_1(\mu, \chi_1, \chi_2) \), there are \( B > 0 \) and \( 0 < \beta \neq \chi_2 - B \) such that

\[
a_{\min} - f(\mu, \chi_1, \chi_2, \beta, B) > \frac{3\epsilon_0}{4}.
\]

By (4.1), we then have

\[
\frac{1}{q} \frac{d}{dt} \int_{\Omega} (u + v)^{-q} \leq -\frac{3\epsilon_0}{4} \int_{\Omega} (u + v)^{-q} + (b_{\max} + c_{\max}) \int_{\Omega} (u + v)^{-q+1}
\]

for all \( t > 0 \), where \( q \) is as in (4.2).
Next, note that if \( q \geq 1 \), by Young’s inequality, there is \( C_{\epsilon_0} > 0 \) such that
\[
(b_{\text{max}} + c_{\text{max}}) \int_\Omega (u + v)^{-q+1} \leq \frac{\epsilon_0}{4} \int_\Omega (u + v)^{-q} + C_{\epsilon_0},
\]
and if \( q < 1 \), then by the Hölder inequality
\[
(b_{\text{max}} + c_{\text{max}}) \int_\Omega (u + v)^{-q+1} \leq (b_{\text{max}} + c_{\text{max}})|\Omega|^q \cdot \left( \int_\Omega u + v \right)^{1-q} \\
\leq (b_{\text{max}} + c_{\text{max}})|\Omega|^q (m^*(\tau, u_0, v_0))^{1-q}
\]
for any \( t > \tau \geq 0 \), where \( m^*(\tau, u_0, v_0) = m_1^*(\tau, u_0, v_0) + m_2^*(\tau, u_0, v_0) \). By (4.12)-(4.14), there is \( \tilde{C}_{\epsilon_0, \tau, u_0, v_0} > 0 \) such that
\[
\frac{1}{q} \frac{d}{dt} \int_\Omega (u + v)^{-q} \leq -\frac{\epsilon_0}{2} \int_\Omega (u + v)^{-q} + \tilde{C}_{\epsilon_0, \tau, u_0, v_0} \quad \forall t > \tau > 0,
\]
where \( \tilde{C}_{\epsilon_0, \tau, u_0, v_0} = C_{\epsilon_0} \) if \( q \geq 1 \) and \( \tilde{C}_{\epsilon_0, \tau, u_0, v_0} = (b_{\text{max}} + c_{\text{max}})|\Omega|^q (m^*(\tau, u_0, v_0))^{1-q} \) if \( q < 1 \). This implies that
\[
\int_\Omega (u(x, \tau; u_0, v_0) + v(x, \tau; u_0, v_0))^{-q} dx \\
\leq e^{-\frac{\epsilon_0 q}{2}(t-\tau)} \int_\Omega (u(x, \tau; u_0, v_0) + v(x, \tau; u_0, v_0))^{-q} dx + 2q \tilde{C}_{\epsilon_0, \tau, u_0, v_0} \epsilon_0^{-1}
\]
for any \( \tau > 0 \) and \( t > \tau \). Theorem 1.4(1) then follows with
\[
M_3 = \begin{cases} C_{\epsilon_0} & \text{if } q \geq 1 \\ 0 & \text{if } q < 1 \end{cases} \quad \text{and} \quad C_{\tau, u_0, v_0} = \begin{cases} 0 & \text{if } q \geq 1 \\ (b_{\text{max}} + c_{\text{max}})|\Omega|^q (m^*(\tau, u_0, v_0))^{1-q} & \text{if } q < 1. \end{cases}
\]

(2) Theorem 1.4(2) follows from the arguments of Theorem 1.2. To be a little more precise, let \( p \) be as in Proposition 3.1. By (3.2) and (3.3), for any \( \epsilon > 0 \), (3.5) holds for all \( t > 0 \). Next, by (3.5) and Proposition 3.1, there are \( 0 < \epsilon_0 < 1 \) and \( C_1, C_2, C_3, C_4 > 0 \) such that (3.6)-(3.9) hold for any \( \tau > 0, t \geq \tau \) and \( \hat{T} = \infty \). In particular, we have the following version of (3.9)
\[
\frac{d}{dt} \int_\Omega (u + v)^p(t, x) dx \leq -\int_\Omega (u + v)^p(t, x) dx \\
+ C_4 \left( \frac{1}{\inf_{\tau \leq f < \infty, x \in \Omega} w_{p+1-\epsilon_0/2}(t, x)} + 1 \right)^{2(p+1)/\epsilon_0} + C_4
\]
for all \( \tau > 0 \) and \( t \geq \tau \). Then by (4.17) and the comparison principle for scalar ODEs, we have

\[
\int_{\Omega} (u + v)^p(t, x) \, dx \leq e^{-\varepsilon_0(t-\tau)} \int_{\Omega} (u + v)^p(\tau, x) \, dx + C_4 \left( \frac{1}{\inf_{\tau \leq r < \infty, x \in \Omega} u^{p+1-\varepsilon_0/2}(t, x)} + 1 \right)^{2(p+1)/\varepsilon_0} + C_4
\]

for any \( \tau > 0 \) and \( t \geq \tau \). Theorem 1.2(1) thus follows with \( M_4 = M_1 := C_4 \). \( \square \)

4.2. Proof of Theorem 1.5

In this subsection, we investigate the ultimate upper bounds of \( \int_{\Omega} (u + v)^{-q}, \int_{\Omega} (u + v)^p, \) and \( \|u + v\|_\infty \) for globally defined classical solutions of (1.2), ultimate lower bound of \( \inf_{\omega \in \Omega} (u + v) \), and prove Theorem 1.5.

**Proof of Theorem 1.5.** (1) First, we prove (1.25). Let \( \varepsilon_0 = a_{\min} - \chi^*(\mu, \chi_1, \chi_2) \). By Theorem 1.4(1), there is \( M_3 > 0 \) such that

\[
\int_{\Omega} (u(x, t; u_0, v_0) + v(x, t; u_0, v_0))^{-q} \, dx \\ \leq e^{-\frac{\varepsilon_0}{2}(t-\tau)} \int_{\Omega} (u(x, \tau; u_0, v_0) + v(x, \tau; u_0, v_0))^{-q} \, dx + M_3 + 2qC_{\tau, u_0, v_0} \varepsilon_0 \tag{4.18}
\]

for all \( t > \tau \), where \( C_{\tau, u_0, v_0} \) is as in (4.16). By Lemma 2.7,

\[
\limsup_{\tau \to \infty} \int_{\Omega} u(\tau, x; u_0, v_0) \leq \frac{a_1|\Omega|}{b_1} \quad \text{and} \quad \limsup_{\tau \to \infty} \int_{\Omega} v(\tau, x; u_0, v_0) \, dx \leq \frac{a_2|\Omega|}{b_2}.
\]

This implies that there is \( \tau > 0 \) such that

\[
C_{\tau, u_0, v_0} \leq 2(b_{\max} + c_{\max})|\Omega|^q \left( \frac{a_1|\Omega|}{b_1} + \frac{a_2|\Omega|}{b_2} \right)^{1-q}.
\]

This together with (4.18) implies that

\[
\limsup_{t \to \infty} \int_{\Omega} (u(t, x; u_0, v_0) + v(t, x; u_0, v_0))^{-q} \leq M^*_1,
\]

where

\[
M^*_1 = M_3 + 2(b_{\max} + c_{\max})|\Omega|^q \left( \frac{a_1|\Omega|}{b_1} + \frac{a_2|\Omega|}{b_2} \right)^{1-q}.
\]

(1.25) is thus proved.
Next, we prove (1.26). By Theorem 1.4(2), for every $\tau > 0$,

$$
\int_{\Omega} (u + v)^p(t, x; u_0, v_0)dx \leq e^{-(t-\tau)} \int_{\Omega} (u + v)^p(\tau, x; u_0, v_0)dx + M_4 \left( \frac{1}{\inf_{\tau \leq t < \infty, x \in \Omega} w^{p + 1 - \varepsilon_0/2}(t, x; u_0, v_0)} + 1 \right)^{2(p+1)/\varepsilon_0} + M_4
$$

for all $t > \tau$. Note that

$$
|\Omega| = \int_{\Omega} (u + v)^{\frac{q}{q+1}} (u + v)^{-\frac{q}{q+1}} dx \leq \left( \int_{\Omega} (u + v)^{\frac{q}{q+1}} \right)^{\frac{q}{q+1}} \left( \int_{\Omega} (u + v)^{-q} \right)^{-\frac{1}{q}}.
$$

This implies that

$$
\int_{\Omega} (u + v) \geq |\Omega|^{\frac{q+1}{q}} \left( \int_{\Omega} (u + v)^{-q} \right)^{-\frac{1}{q}}.
$$

Then by Lemma 2.4,

$$
w(t, x) \geq \delta_0 \int_{\Omega} (u(t, x) + v(t, x)) dx \geq \delta_0 |\Omega|^{\frac{q+1}{q}} \left( \int_{\Omega} (u + v)^{-q} (t, x) dx \right)^{-\frac{1}{q}}
$$

for all $t > 0$ and $x \in \Omega$. This implies that

$$
\inf_{\tau \leq t < \infty, x \in \Omega} \frac{1}{w^{p + 1 - \varepsilon_0/2}(t, x)} = \sup_{\tau \leq t < \infty, x \in \Omega} \frac{1}{w^{p + 1 - \varepsilon_0/2}(t, x)} \leq \left( \sup_{\tau \leq t < \infty, x \in \Omega} \frac{(\int_{\Omega} (u + v)^{-q} (t, x) dx)^{\frac{1}{q}}}{\delta_0 |\Omega|^{\frac{q+1}{q}}} \right)^{p + 1 - \varepsilon_0/2}.
$$

By (1.25), there is $\tau > 0$ such that

$$
\inf_{\tau \leq t < \infty, x \in \Omega} \frac{1}{w^{p + 1 - \varepsilon_0/2}(t, x)} \leq \left( \frac{2M_1^*}{{\delta_0 |\Omega|^{\frac{q+1}{q}}} \varepsilon_0} \right)^{p + 1 - \varepsilon_0/2}.
$$

This together with (4.19) implies that

$$
\limsup_{t \to \infty} \int_{\Omega} (u(t, x) + v(t, x))^p \leq M_2^*,
$$

where
\[ M_2^* = M_4 \left( \left( \frac{2M_1^*}{{\delta_0}|\Omega|^q} \right)^{p+1} + 1 \right)^{2(p+1)/\varepsilon_0} + M_4. \]

(1.26) is thus proved.

Next, we prove (1.27) by the arguments of Theorem 1.2(2). First, for given \( p > 2N \) and \( 0 < \theta < 1 - \frac{2N}{p} \), choose \( \beta \in (0, \frac{1}{2}) \) such that \( 2\beta - \frac{2N}{p} > \theta \). By Lemma 2.2, there is \( C_{p,\beta} > 0 \) such that for any \( u_0, v_0 \) satisfying (1.11), (3.10) hold for all \( t > 0 \), that is,

\[ \|u(\cdot, \cdot) + v(\cdot, \cdot)\|_{C^0(\hat{\Omega})} \leq C_{p,\beta}\|u(t, \cdot) + v(t, \cdot)\|_{X_p^\beta} \quad \forall 0 < t < \infty, \tag{4.21} \]

and (3.11) holds for any \( \tau > 0 \) and \( t > \tau \), that is,

\[ \begin{align*}
\|u(t, \cdot) + v(t, \cdot)\|_{X_p^\beta} &\leq \|A_p^{\beta}e^{-A_p^p(t-s)}(u(\tau, \cdot; u_0, 0) + v(\tau, \cdot; u_0, 0))\|_{L^p/2(\Omega)} \\
&+ \int_\tau^t \|A_p^{\beta}e^{-A_p^p(t-s)}\nabla \cdot \left( \frac{\chi_1u(s, \cdot) + \chi_2v(s, \cdot)}{w(s, \cdot)} \nabla w(s, \cdot) \right)\|_{L^p/2(\Omega)} ds \\
&+ \int_\tau^t \|A_p^{\beta}e^{-A_p^p(t-s)}g(u(s, \cdot), v(s, \cdot))\|_{L^p/2(\Omega)} ds
\end{align*} \tag{4.22} \]

for any \( t > \tau > 0 \), where

\[ g(u, v) = (\mu + a_1)u + (\mu + a_2)v - (b_1u^2 + b_2v^2) - (c_1 + c_2)uv. \]

Next, by Lemma 2.2, there is \( \gamma > 0 \) such that (3.12) holds for any \( \tau > 0 \) and \( t > \tau \), that is,

\[ \begin{align*}
\|A_p^{\beta}e^{-A_p^p(t-s)}(u(\tau, \cdot; u_0, 0) + v(\tau, \cdot; u_0, 0))\|_{L^p/2(\Omega)} \\
&\leq C_{p,\beta}(t - \tau)^{-\beta}e^{-\gamma(t - \tau)}\|u(\tau, \cdot; u_0, 0) + v(\tau, \cdot; u_0, 0)\|_{L^p/2} \quad \forall 0 < \tau < t < \infty. \tag{4.23}
\end{align*} \]

By Lemma 2.3 and Lemma 2.5, for any \( \varepsilon > 0 \), there are \( \gamma > 0 \), \( C_p > 0 \) and \( C_{p,\beta,\varepsilon} > 0 \) such that (3.13) with \( \inf_{t \leq T, x \in \Omega} w(t, x) \) being replaced by \( \inf_{t \leq \infty, x \in \Omega} w(t, x) \) holds for any \( \tau > 0 \) and \( t > \tau \), that is,

\[ \begin{align*}
\int_\tau^t \|A_p^{\beta}e^{-A_p^p(t-s)}\nabla \cdot \left( \frac{\chi_1u(s, \cdot) + \chi_2v(s, \cdot)}{w(s, \cdot)} \nabla w(s, \cdot) \right)\|_{L^p/2(\Omega)} ds \\
&\leq C_pC_{p,\beta,\varepsilon}(\chi_1 + \chi_2) \int_\tau^t (1 + (t - s)^{-\beta - \frac{1}{2} - \varepsilon})e^{-\gamma(t - s)}\|u(s, \cdot) + v(s, \cdot)\|_{L^p(\Omega)}^2 ds \tag{4.24}
\end{align*} \]

for any \( t > \tau > 0 \). It then follows that
for any $t > \tau > 0$, where $M_2$ is as in Theorem 1.2(2). This together with (1.25), (1.26), and (4.20) implies that

$$\limsup_{t \to \infty} \|u(t, \cdot; u_0, v_0) + v(t, \cdot; u_0, v_0)\|_{C^0(\bar{\Omega})} \leq M^*_3,$$

where

$$M^*_3 = M_2 \left( \frac{2M^*_1 M^*_2}{\delta_0 \Omega^{\frac{q+1}{q}}} + M^*_2 + 1 \right).$$

(1.27) is thus proved.

(2) We prove it by contradiction. Assume that there is no $M^*_0 > 0$ such that (1.28) holds. Then for any $n \in \mathbb{N}$, there are $u_n, v_n$ satisfying (1.11) such that

$$\liminf_{t \to \infty} \inf_{x \in \Omega} (u(t, x; 0, u_n, v_n) + v(t, x; 0, u_n, v_n)) \leq m_n := \frac{1}{n}.$$

By Theorem 1.4(1) and (4.20), there are $T_n > 0$ and $\tilde{M}^*_1$ such that

$$\int_{\Omega} u(t, x; u_n, v_n) + v(t, x; u_n, v_n)dx \geq \tilde{M}^*_1 \quad \forall t \geq T_n$$

and

$$\|u(t, x; u_n, v_n) + v(t, x; u_n, v_n)\|_{C^0(\bar{\Omega})} \leq 2M^*_3 \quad \forall t \geq T_n.$$

By (4.26), there are $t_n \in \mathbb{R}$ with $t_n \geq T_n + 1$ and $x_n \in \bar{\Omega}$ such that

$$u(t_n, x_n; u_n, v_n) + v(t_n, x_n; u_n, v_n) \leq \frac{2}{n}.$$  

Note that

$$u(t_n, x; u_n, v_n) = u(1, x; u(t_n - 1, \cdot; u_n, v_n), v(t_n - 1, u_n, v_n))$$

and
\[
v(t_n, x; 0, u_n, v_n) = v(1, x; u(t_n - 1, \cdot; u_n, v_n), v(t_n - 1, \cdot; u_n, v_n)). \tag{4.31}
\]

By (4.28), without loss of generality, we may assume that there are \(u_0^*, v_0^*, u_1^*, v_1^* \in C(\overline{\Omega})\) such that
\[
\lim_{n \to \infty} u(t_n - 1, x; u_n, v_n) = u_0^*(x), \quad \lim_{n \to \infty} v(t_n - 1, x; u_n, v_n) = v_0^*(x),
\]
and
\[
\lim_{n \to \infty} u(t_n, x; u_n, v_n) = u_1^*(x), \quad \lim_{n \to \infty} v(t_n, x; u_n, v_n) = v_1^*(x).
\]

uniformly in \(x \in \overline{\Omega}\). Then, by (4.30) and (4.31), we have that
\[
u'_1(x) + v'_1(x) = u(1, x; u_0^*, v_0^*) + v(1, x; u_0^*, v_0^*). \tag{4.32}\]

By (4.27), (4.28), and the Dominated Convergence Theorem,
\[
\int_{\Omega} u_0^*(x) + v_0^*(x) = \lim_{n \to \infty} \int_{\Omega} (u(t_n - 1, x; 0, u_n, v_n) + v(t_n - 1, x; 0, u_n, v_n))dx \geq \tilde{M}_1^*.
\]

Then by (4.32) and comparison principles for parabolic equations,
\[
\inf_{x \in \Omega} (u_1^*(x) + v_1^*(x)) > 0.
\]

This together with (4.29) implies that
\[
\frac{2}{n} \geq \inf_{x \in \Omega} (u(t_n, x; 0, u_n, v_n) + v(t_n, x; 0, u_n, v_n)) \geq \frac{1}{2} \inf_{x \in \Omega} (u_1^*(x) + v_1^*(x)) \quad \forall n \gg 1,
\]
which is a contradiction. Therefore (2) holds. \(\square\)

5. Appendix: proofs of Propositions 3.1 and 3.2

In this section, we outline the proofs of Propositions 3.1 and 3.2. Throughout this section, for given \(u_0, v_0\) satisfying (1.11), we put
\[
(u(t, x), v(t, x), w(t, x)) := (u(t, x; u_0, v_0), v(t, x; u_0, v_0), w(t, x; u_0, v_0)),
\]
and if no confusion occurs, we may drop \((t, x)\) in \(u(t, x)\) (resp. \(v(t, x), w(t, x)\)).
5.1. Outline of the proof of Proposition 3.1

Proposition 3.1 can be proved by the similar arguments as those in [26, Proposition 4.1]. For the reader’s convenience, we provide some outline of the proof of Proposition 3.1 in the following.

Outline of the proof of Proposition 3.1. First of all, observe that if $c, d, r, l > 0$ are positive constants such that $cd - c - d > 0$ and $p - l - r - 1 > 0$, then by Young’s inequality, we have

\[
\int_{\Omega} \frac{(u + v)^{p-1}}{w} \nabla(u + v) \cdot \nabla w = \int_{\Omega} (u + v)^l \nabla(u + v) \cdot (u + v)^r \frac{\nabla w}{w} \cdot (u + v)^{p-l-r-1} \\
\leq \frac{1}{c} \int_{\Omega} (u + v)^l |\nabla(u + v)|^c + \frac{1}{d} \int_{\Omega} (u + v)^r d \left(\frac{\nabla w}{w}\right)^d \\
+ \frac{cd - c - d}{cd} \int_{\Omega} ((u + v)^{p-l-r-1}) \frac{cd}{cd - c - d},
\]

(5.1)

Next, by [26, Lemmas 4.1, 4.2], there are positive numbers $1 < c < 2, d, l, r > 0$, and $p > 3N$ such that the following hold,

\[
p > \max\{3N, l + r + 1\}, \tag{5.2}
\]
\[
md - c - d > 0, \tag{5.3}
\]
\[
m := \frac{(2l - p + 2)c}{2} > 0, \tag{5.4}
\]
\[
\frac{2m}{2 - c} < p + 1, \tag{5.5}
\]
\[
0 < \frac{d(p+1)}{p+1-r} < 2p + 2, \tag{5.6}
\]

and

\[
\frac{cd(p-l-r-1)}{cd - c - d} < p + 1. \tag{5.7}
\]

For any $0 < \varepsilon_0 \ll 1$, we then have

\[
p + 1 - \varepsilon_0 > \max\left\{\frac{2m}{2 - c}, rd, \frac{cd(p-l-r-1)}{cd - c - d}\right\} \tag{5.8}
\]

and

\[
2p + 2 - \varepsilon_0 > \frac{d(p+1-\varepsilon_0)}{p+1-\varepsilon_0 - rd}. \tag{5.9}
\]
Now, for the above \( c, d, l, r, p \) and \( m \), we have \( m > 0 \) and \( l c - m > 0 \). Then by Young’s inequality, for any \( C_0 > 0 \),

\[
I_1 = \int_{\Omega} (u + v)^{l c} |\nabla (u + v)|^c \\
= \int_{\Omega} (u + v)^{l c} \cdot (u + v)^{l c - m} |\nabla (u + v)|^c \\
\leq \frac{2 - c}{2} \left( \frac{1}{2 C_0} \right)^{\frac{c}{2 - c}} \int_{\Omega} (u + v)^{l c - m} + c C_0 \int_{\Omega} (u + v)^{l c - m} |\nabla (u + v)|^c \\
= \frac{2 - c}{2} \left( \frac{1}{2 C_0} \right)^{\frac{c}{2 - c}} \int_{\Omega} (u + v)^{l c - m} + c C_0 \int_{\Omega} (u + v)^{2l - 2m} |\nabla (u + v)|^2
\]

for \( t \in (0, T_{\text{max}} (u_0, v_0)) \). By the definition of \( m \), \( 2l - \frac{2m}{c} = p - 2 \). By (5.8) and Young’s inequality again, we have

\[
I_1 \leq C + C \int_{\Omega} (u + v)^{p + 1 - \varepsilon_0} + C C_0 \int_{\Omega} (u + v)^{p - 2} |\nabla (u + v)|^2
\]

for \( t \in (0, T_{\text{max}} (u_0, v_0)) \) and some \( C > 0 \) (independent of \( u \) and \( v \)).

Applying Young’s inequality, we have that

\[
I_2 = \int_{\Omega} (u + v)^{r d} \left( \frac{|\nabla w|}{w} \right)^{d} \\
\leq \frac{r d}{p + 1 - \varepsilon_0} \int_{\Omega} ((u + v)^{r d} + p + 1 - \varepsilon_0 - r d \int_{\Omega} \left( \frac{|\nabla w|}{w} \right)^{d \cdot \frac{p + 1 - \varepsilon_0}{p + 1 - \varepsilon_0 - r d}} \\
= \frac{r d}{p + 1 - \varepsilon_0} \int_{\Omega} (u + v)^{p + 1 - \varepsilon_0} + \frac{p + 1 - \varepsilon_0 - r d}{p + 1 - \varepsilon} \int_{\Omega} \left( \frac{|\nabla w|}{w} \right)^{d \cdot \frac{p + 1 - \varepsilon_0}{p + 1 - \varepsilon_0 - r d}}
\]

for \( t \in (0, T_{\text{max}} (u_0, v_0)) \). Then by (5.9) and Young’s inequality again,

\[
I_2 \leq C \int_{\Omega} (u + v)^{p + 1 - \varepsilon_0} + C \int_{\Omega} \left( \frac{|\nabla w|}{w} \right)^{2p + 2 - \varepsilon_0} + C
\]

for \( t \in (0, T_{\text{max}} (u_0, v_0)) \) and some \( C > 0 \) (independent of \( u \) and \( v \)).
By (5.8) and Young’s inequality, we have
\[ I_3 = \int_{\Omega} ((u + v)^{p-l-r-1}) \frac{cd}{cd-c-d} \]
\[ = \int_{\Omega} (u + v)^{\frac{cd(p-l-r-1)}{cd-c-d}} \]
\[ \leq \frac{cd(p-l-r-1)}{(p+1-\varepsilon_0)(cd-c-d)} \int_{\Omega} (u + v)^{p+1-\varepsilon_0} + C \]
for \( t \in (0, T_{\text{max}}(u_0, v_0)) \) and some \( C > 0 \) (independent of \( u \) and \( v \)). The proposition then follows from (5.1) and the above estimates for \( I_1, I_2, \) and \( I_3 \). \( \square \)

5.2. Outline of the proof of Proposition 3.2

Proposition 3.2 can be proved by the similar arguments as those in [26, Proposition 4.2]. For the reader’s convenience, we provide a proof of Proposition 3.2 in the following.

We first present some lemmas.

Lemma 5.1. Let \( p \geq 3 \) and \( k \geq 2 \). Then
\[ (k-1) \int_{\Omega} \frac{|\nabla w|^2}{w^k} \leq (p-1) \int_{\Omega} \frac{|\nabla w|^{2p-4}}{w^{k-1}} \nabla w \cdot \nabla |\nabla w|^2 + \mu \int_{\Omega} \frac{|\nabla w|^{2p-2}}{w^{k-2}} \] (5.10)
for all \( t \in (0, T_{\text{max}}(u_0, v_0)) \).

Proof. Multiplying the third equation in (1.2) by \( \frac{|\nabla w|^{2p-2}}{w^{k-2}} \) and integrating it over \( \Omega \) yields that
\[ (k-1) \int_{\Omega} \frac{|\nabla w|^2}{w^k} + v \int_{\Omega} \frac{|\nabla w|^{2p-4}}{w^{k-1}} \nabla w \cdot \nabla |\nabla w|^2 + \lambda \int_{\Omega} \frac{|\nabla w|^{2p-2}}{w^{k-1}} \]
\[ = (p-1) \int_{\Omega} \frac{|\nabla w|^{2p-4}}{w^{k-1}} \nabla w \cdot \nabla |\nabla w|^2 + \mu \int_{\Omega} \frac{|\nabla w|^{2p-2}}{w^{k-2}} \]
for all \( t \in (0, T_{\text{max}}(u_0, v_0)) \). This implies (5.10). \( \square \)

To prove Proposition 3.2, it is then essential to provide proper estimate for the integral \( \int_{\Omega} \frac{|\nabla w|^{2p-4}}{w^{k-1}} \nabla w \cdot \nabla |\nabla w|^2 \). By Young’s inequality, for any \( 0 < \varepsilon < p - \frac{3}{2} \), we have
\[ \int_{\Omega} \frac{|\nabla w|^{2p-4}}{w^{k-1}} \nabla w \cdot \nabla |\nabla w|^2 \leq \frac{p - \frac{3}{2} - \varepsilon}{p + k - 3} \int_{\Omega} \frac{|\nabla w|^{2p-6}}{w^{k-2}} |\nabla |\nabla w|^2|^2 \]
By a direct calculation, we have

\[
(p - 2) \int_\Omega \frac{1}{w^{k-2}} |\nabla |\nabla w| |^2\, d\Omega = \int_\Omega \frac{1}{w^{k-2}} \frac{\partial |\nabla w|^2}{\partial v} + (k - 2) \int_\Omega \frac{1}{w^{k-1}} \nabla w \cdot \nabla |\nabla w|^2 \\
\quad \quad \quad - \int_\Omega \frac{1}{w^{k-2}} \Delta |\nabla w|^2.
\] (5.12)

Note that

\[
\Delta |\nabla w|^2 = 2 \nabla w \cdot \nabla (\Delta w) + 2 |D^2 w|^2 = 2 \nabla w \cdot (\nabla (\mu w - \nu u - \lambda v) + 2 |D^2 w|^2,
\]

and

\[
|\nabla |\nabla w| |^2 = 2 \sum_{i=1}^n \nabla w \cdot \nabla w_{x_i} |^2 \leq 4 |\nabla w|^2 \sum_{i=1}^n |\nabla w_{x_i} |^2 = 4 |\nabla w|^2 |D^2 w|^2.
\]

It then follows that

\[
- \int_\Omega \frac{1}{w^{k-2}} \Delta |\nabla w|^2 \leq 2 \int_\Omega \frac{1}{w^{k-2}} \nabla (\nu u + \lambda v) \cdot \nabla w \\
- 2 \mu \int_\Omega \frac{1}{w^{k-2}} |\nabla w|^2 - \frac{1}{2} \int_\Omega \frac{1}{w^{k-2}} |\nabla |\nabla w| |^2|.
\] (5.13)

By (5.12) and (5.13), we have

\[
(p - \frac{3}{2}) \int_\Omega \frac{1}{w^{k-2}} |\nabla w|^2 \, d\Omega \leq \int_\Omega \frac{1}{w^{k-2}} \frac{\partial |\nabla w|^2}{\partial v} + (k - 2) \int_\Omega \frac{1}{w^{k-1}} \nabla w \cdot \nabla |\nabla w|^2 \\
\quad \quad \quad + 2 \int_\Omega \frac{1}{w^{k-2}} \nabla (\nu u + \lambda v) \cdot \nabla w - 2 \mu \int_\Omega \frac{1}{w^{k-2}} |\nabla w|^2.
\] (5.14)

To get proper estimate for \( \int_\Omega \frac{|\nabla w|^{2p-4}}{w^{k-2}} \nabla w \cdot \nabla |\nabla w|^2 \), it is then essential to provide proper estimates for the integrals \( \int_{\partial \Omega} \frac{|\nabla w|^{2p-4}}{w^{k-2}} \frac{\partial |\nabla w|^2}{\partial v} \) and \( \int_\Omega \frac{|\nabla w|^{2p-4}}{w^{k-2}} \nabla (\nu u + \lambda v) \cdot \nabla w \).

**Lemma 5.2.** For every \( \epsilon > 0 \), there is \( C > 0 \) such that

\[
\int_{\partial \Omega} \frac{|\nabla w|^{2p-4}}{w^{k-2}} \frac{\partial |\nabla w|^2}{\partial v} \leq \epsilon \int_\Omega \frac{|\nabla w|^{2p-6}}{w^{k-2}} |\nabla |\nabla w| |^2| + C \int_\Omega \frac{|\nabla w|^{2p-2}}{w^{k-2}}
\]

290
for all \( t \in (0, T_{\text{max}}(u_0, v_0)) \).

**Proof.** It follows from the similar arguments in [26, Lemma 4.4]. \( \square \)

**Lemma 5.3.** Let \( p \geq 3 \) and \( k \geq p \). For every given \( \varepsilon > 0 \), there is \( M > 0 \) such that

\[
\int_{\Omega} |\nabla w|^{2p-4} w^{k-2} \nabla (vu + \lambda v) \cdot \nabla w \leq M \int_{\Omega} \frac{(vu + \lambda v)^p}{w^{k-p}} + \varepsilon \int_{\Omega} |\nabla w|^{2p-6} w^{k-2} |\nabla |\nabla w|^2|^2 + \varepsilon \int_{\Omega} \frac{|\nabla w|^{2p}}{w^k}
\]

for all \( t \in (0, T_{\text{max}}(u_0, v_0)) \).

**Proof.** First, we have that

\[
\int_{\Omega} \frac{|\nabla w|^{2p-4}}{w^{k-2}} \nabla (vu + \lambda v) \cdot \nabla w = (k - 2) \int_{\Omega} \frac{(vu + \lambda v)|\nabla w|^{2p-2}}{w^{k-1}} - \int_{\Omega} \frac{(vu + \lambda v)|\nabla w|^{2p-4}}{w^{k-2}} \Delta w
\]

\[
= (k - 2) \int_{\Omega} \frac{(vu + \lambda v)|\nabla w|^{2p-2}}{w^{k-1}} - (p - 2) \int_{\Omega} \frac{(vu + \lambda v)|\nabla w|^{2p-6}}{w^{k-2}} \nabla w \cdot \nabla |\nabla w|^2.
\]

Next, by Young’s inequality, for every \( B_1 > 0 \), there exists a positive constant \( A_1 = A_1(k, p, B_1) > 0 \) such that

\[
I_1 = (k - 2) \int_{\Omega} \frac{(vu + \lambda v)|\nabla w|^{2p-2}}{w^{k-1}} = (k - 2) \int_{\Omega} \frac{vu + \lambda v}{w^{k-p}} \cdot \frac{|\nabla w|^{2p-2}}{w^{(p-1)}}
\]

\[
\leq A_1 \int_{\Omega} \frac{(vu + \lambda v)^p}{w^{k-p}} + B_1 \int_{\Omega} \frac{|\nabla w|^{2p}}{w^k}. \quad (5.15)
\]

Now, by the fact that \( \Delta w = \mu w - vu - \lambda v \), and Young’s inequality, for every \( B_2 > 0 \), there exists a positive constant \( A_2 = A_2(k, p, v, \lambda, B_2) > 0 \) such that

\[
-I_2 = - \int_{\Omega} \frac{(vu + \lambda v)|\nabla w|^{2p-4}}{w^{k-2}} \Delta w
\]

\[
= -\mu \int_{\Omega} \frac{(vu + \lambda v)|\nabla w|^{2p-4}}{w^{k-3}} + \int_{\Omega} \frac{(vu + \lambda v)^2|\nabla w|^{2p-4}}{w^{k-2}}
\]

\[
\leq \int_{\Omega} \frac{(vu + \lambda v)^2}{w^{2(k-2) - p}} \cdot \frac{|\nabla w|^{2p-4}}{w^{(p-2)}}
\]
Finally, for every $\varepsilon > 0$ and $B_3 > 0$, there are positive constants $A_3 = A_3(\varepsilon, k, p, \nu, B_3) > 0$ and $A_4 = A_4(A_4, B_3)$ such that

\begin{equation}
-I_3 = -(p - 2) \int_{\Omega} \frac{(\nu u + \lambda v)|\nabla w|^{2p-6}}{w^{k-2}} \nabla w \cdot \nabla |\nabla w|^2
\leq A_3 \int_{\Omega} \frac{(\nu u + \lambda v) |\nabla w|^{2p-4}}{w^{k-2}} + \varepsilon \int_{\Omega} |\nabla w|^{2p-6} \frac{|\nabla |\nabla w|^2|^2}{w^{k-2}}
\leq A_4 \int_{\Omega} \frac{(\nu u + \lambda v) p}{w^{k-p}} + B_3 \int_{\Omega} |\nabla w|^{2p} w^{k-p} + \varepsilon \int_{\Omega} |\nabla w|^{2p-6} \frac{|\nabla |\nabla w|^2|^2}{w^{k-2}}.
\end{equation}

Combining (5.15), (5.16) and (5.17) with $B_1 = B_2 = B_3 = \frac{1}{3} \varepsilon$ and $M = M(\varepsilon, k, p, \nu) := A_1 + A_2 + A_4$ completes the proof. □

We now prove Proposition 3.2.

**Proof of Proposition 3.2.** First, by (5.14) and Lemmas 5.2 and 5.3, for any $0 < \varepsilon < p - \frac{3}{2}$, there are $M, C > 0$ such that

\begin{equation}
\left( p - \frac{3}{2} \right) \int_{\Omega} \frac{|\nabla w|^{2p-6}}{w^{k-2}} |\nabla |\nabla w|^2|^2 \leq \varepsilon \int_{\Omega} |\nabla w|^{2p-6} \frac{|\nabla |\nabla w|^2|^2}{w^{k-2}} + \varepsilon \int_{\Omega} |\nabla w|^{2p} w^{k-p}
+ (k - 2) \int_{\Omega} \frac{|\nabla w|^{2p-4}}{w^{k-1}} \nabla w \cdot \nabla |\nabla w|^2
+ M \int_{\Omega} \frac{(\nu u + \lambda v) p}{w^{k-p}} + C \int_{\Omega} \frac{|\nabla w|^{2p-2}}{w^{k-2}}.
\end{equation}

Next, by (5.11) and (5.18), we have

\begin{equation}
\frac{p - 1}{p + k - 3} \int_{\Omega} \frac{|\nabla w|^{2p-4}}{w^{k-1}} \nabla w \cdot \nabla |\nabla w|^2
\leq \frac{\varepsilon}{p + k - 3} \int_{\Omega} \frac{|\nabla w|^{2p}}{w^{k}} + \frac{M}{p + k - 3} \int_{\Omega} \frac{(\nu u + \lambda v) p}{w^{k-p}}
+ \frac{C}{p + k - 3} \int_{\Omega} \frac{|\nabla w|^{2p-2}}{w^{k-2}} + \frac{p + k - 3}{4 (p - \frac{3}{2} - \varepsilon)} \int_{\Omega} \frac{|\nabla w|^{2p}}{w^{k}}.
\end{equation}
Now, by (5.10), and (5.19), for any \( 0 < \varepsilon < p - \frac{3}{2} \), there holds

\[
\left( k - 1 - \frac{(p + k - 3)^2}{4(p - \frac{3}{2} - \varepsilon)} - \varepsilon \right) \int_{\Omega} \frac{|\nabla w|^{2p}}{w^k} \leq M \int_{\Omega} \frac{(\nu u + \lambda v)^p}{w^{k-p}} + (C + \mu) \int_{\Omega} \frac{|\nabla w|^{2p-2}}{w^{k-2}}.
\] (5.20)

By Young’s inequality, for any \( 0 < \varepsilon < p - \frac{3}{2} \), there is \( \tilde{M}(\varepsilon, p) \) such that

\[
(C + \mu) \int_{\Omega} \frac{|\nabla w|^{2p-2}}{w^{k-2}} \leq \varepsilon \int_{\Omega} \frac{|\nabla w|^{2p}}{w^k} + \tilde{M} \int_{\Omega} w^{2p-k}.
\] (5.21)

By (5.20) and (5.21),

\[
\left( k - 1 - \frac{(p + k - 3)^2}{4(p - \frac{3}{2} - \varepsilon)} - 2\varepsilon \right) \int_{\Omega} \frac{|\nabla w|^{2p}}{w^k} \leq M \int_{\Omega} \frac{(\nu u + \lambda v)^p}{w^{k-p}} + \tilde{M} \int_{\Omega} w^{2p-k}.
\] (5.22)

Finally, since \( p - \sqrt{2p-3} \leq k < p + \sqrt{2p-3} \), one can find \( \varepsilon > 0 \) such that

\[
k - 1 - 2\varepsilon - \frac{(p + k - 3)^2}{4(p - \frac{3}{2} - \varepsilon)} > 0.
\]

The proposition then follows from (5.22).

Data availability

No data was used for the research described in the article.

Acknowledgment

The authors would like to thank the anonymous referee for carefully reading the manuscript and for valuable comments and suggestions which considerably improved the presentation of the paper.

References


