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Shape Optimization for Noise Radiation Problems

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Abstract—A shape design model that reduces the amount of noise radiated from aircraft turbofan engines is studied in this paper. The model is formulated as shape control of the Helmholtz equation with radiation boundary conditions on part of the boundary and incoming waves specified as the source. Existence of optimal shape is proved to show that the model is appropriately established. A numerical experiment is conducted to demonstrate the efficiency of the model. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Optimal control, Shape control, Helmholtz equation.

1. INTRODUCTION

Increasingly stringent noise regulations at airports and demands for more passenger comfort have prompted an increase in the amount of research devoted to aircraft noise reduction and control. A major component of aircraft noise for the large commercial aircraft currently in service is the tone noise generated by the rotating blades of the fan, as they interact with the stator blades and the struts. For a given fan geometry, the propagation inside the nacelle and radiation into free space of this noise component can be modeled numerically, once the source of noise is assumed known. Several methods have been devised towards this end, both in the frequency [1,2] and in the time [3] domain.

In general, passive noise control through laminar acoustic liners is used for fan noise reduction. As requirements become more stringent, other possibilities become worthy of being investigated. This paper addresses the related issue of shape optimization in the frequency domain. Our goal is to find an appropriate shape for the fan inlet so that the amount of noise radiated to the far field from the system is minimized. Shape control problems have been studied extensively in the past 30 years. The mathematical aspect of the problem was studied, among others, by Chenais [4]. He gives the conditions necessary for the existence of an optimal shape for systems governed by coercive elliptic partial differential equations. More recently, minimization of viscous drag was studied through shape modifications in [5]. There has also been extensive research work done on shape identification for acoustic scattering problems [6]. The intrinsic difficulties that

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are encountered in such an approach for the noise radiation problem are related not only to the complex partial differential equations that occur in the presence of a mean flow, but also to the wide range of modes over which the optimization has to be considered for a given frequency, while satisfying constraints that are related to the aerodynamic properties of the inlet shape. As a first study in this direction, this paper considers the Helmholtz equation only, which governs sound radiation in a quiescent fluid, while keeping the possibility to study the simultaneous radiation of several modes.

The work is organized as follows. The next section introduces the mathematical formulation of the optimal control problem under consideration. Section 3 presents the results regarding the existence of the optimal shape. The existence result helps to demonstrate that the state equation and the cost function we set up in Section 2 is appropriate. After that, a numerical discretization using spectral elements is described, and numerical results to support the analysis are given. Some concluding remarks end this paper.

2. THE SHAPE CONTROL PROBLEM

In order to simplify the presentation, the fan is supposed to have an axisymmetric geometry. The domain in which the control problem is formulated has, in this case, the generic shape represented in Figure 1. The modal composition of the noise source is supposed to be known on the source plane Γ_f. The fan inlet casing is made up of two parts, the first part being the boundary Γ_α which spans only the interior of the casing starting from the source plane position $x_1 = a$ and extending to $x_2 = b$. The shape of this boundary is determined by the function $y = \alpha(x)$, and we assume a hard wall boundary condition along it. The other part of the inlet geometry constitutes the boundary Γ_c, along the exterior of the casing with eventually some extent along the interior as well. For the sake of generality, we allow for the presence of an acoustic liner on this boundary. The boundary Γ_∞ is assumed to be sufficiently far from the noise source such that the radiated field behaves locally as a plane wave at local incidence and the Sommerfeld radiation boundary condition holds. The fan symmetry axis is denoted by Γ_a.

The acoustic velocity potential u is supposed to satisfy the Helmholtz equation

$$\Delta u + k^2 u = 0, \qquad \text{on } \Omega, \tag{2.1}$$

subject to the following boundary conditions on the boundary $\partial\Omega$ of Ω:

$$\begin{aligned} u|_{\Gamma_f} &= g(\alpha), \\ \left.\frac{\partial u}{\partial n}\right|_{\Gamma_\alpha} &= 0, \\ \left.\frac{\partial u}{\partial n}\right|_{\Gamma_a} &= 0, \\ \left.\left(u + \chi \frac{\partial u}{\partial n}\right)\right|_{\Gamma_c} &= 0, \\ \left.\left(iku + \frac{\partial u}{\partial n}\right)\right|_{\Gamma_\infty} &= 0, \end{aligned} \tag{2.2}$$

where $\chi > 0$. Both the dependent and the independent variables appearing in the above equations are supposed to be properly nondimensionalized.

The problem consists in manipulating the boundary Γ_α so that the least amount of noise propagates to the far field, while respecting some constraints on the boundary shape. More specifically, we want to find $\alpha(x)$ so that the cost functional

$$J = A \int_\Omega u^2 \, d\Omega + B \int_\Omega |\nabla u|^2 \, d\Omega + \lambda \int_{x_1}^{x_2} (\alpha(x) - \alpha_0(x))^2 \, dx \tag{2.3}$$

is minimized. Here A, B, and λ are three constants and $A^2 + B^2 > 0$, $\lambda \geq 0$.

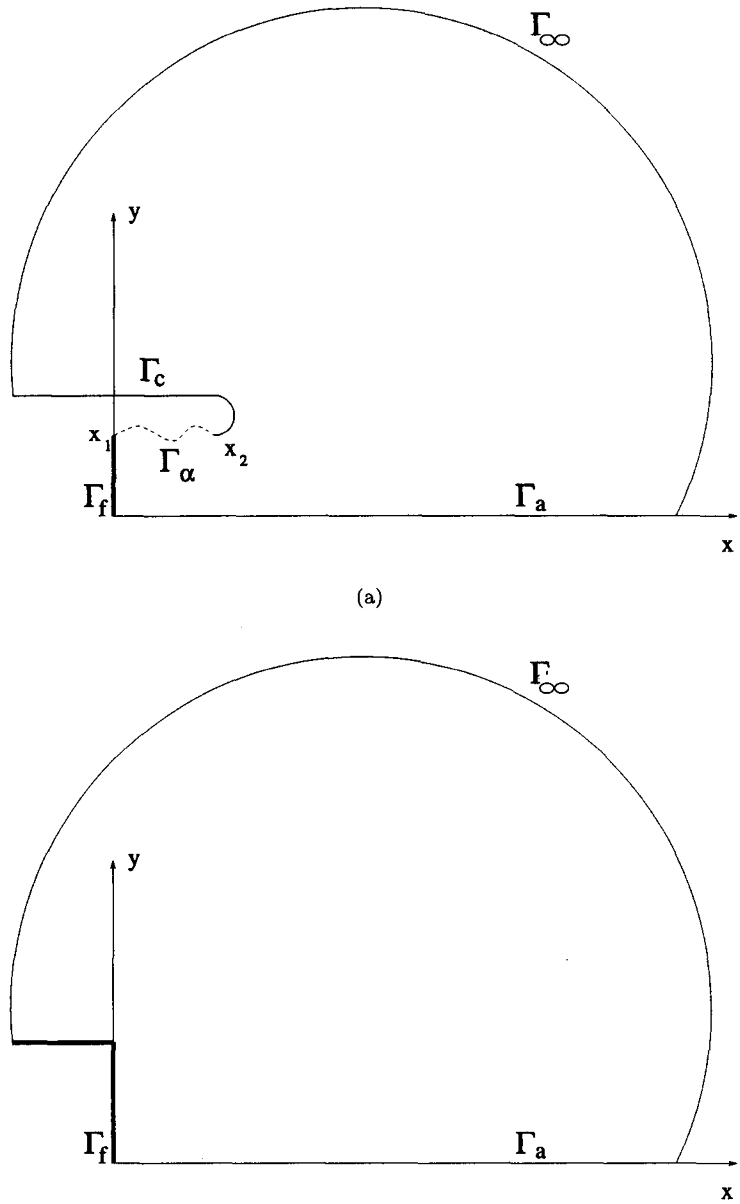


Figure 1. Generic solution domain for the noise radiation problem (a) and the fixed domain $\hat{\Omega}$ (b).

Several comments are necessary for completeness. The constraints on the shape are introduced through a penalty term in J, with α_0 denoting a target shape obtained in a previous design step, most probably based on aerodynamic considerations. The Dirichlet boundary condition on Γ_f does not remain constant during the optimization process. Indeed, $g(\alpha) = g^+ + g^-(\alpha)$, where g^+ denotes the sum of the incoming, positive-x propagating modes, which specifies the sound source and is among the given data of the problem, hence, fixed during optimization. On the other hand, the term $g^-(\alpha)$ represents the sum of the reflected, negative-x propagating modes, that will, in general, depend on the shape of the boundary. The acoustic energy transported by these modes will have as an upper bound the acoustic energy of the incoming modes, as there is no other sound source in the domain. The presence of $g(\alpha)$ makes the problem nonlinear.

The rest of this section introduces some notations that will be used throughout the paper. Let $L^2(\Omega)$ be the usual square integrable function space on Ω. For $u, v \in L^2(\Omega)$, let

$$(u,v) := \int_\Omega u\bar{v}\, d\Omega$$

denote the inner product of the space. For $\Gamma \subset \partial\Omega$, the inner product in $L^2(\Gamma)$ is defined as

$$\langle u,v\rangle := \int_\Gamma u\bar{v}\, d\Gamma.$$

For a given integer m, define $H^m(\Omega)$ as

$$H^m(\Omega) := \left\{u \in L^2(\Omega),\ \partial^\gamma u \in L^2(\Omega),\ |\gamma| \le m\right\}.$$

The norm in $H^m(\Omega)$ is defined as

$$\|u\|_{m,\Omega} := \left(\sum_{|\gamma|\le m} (\partial^\gamma u, \partial^\gamma u)\right)^{1/2}.$$

As a final remark, according to the discussion above, we may assume that the following holds:

$$\|g(\alpha)\|_{H^{1/2}(\Gamma_f)} \le \left\|g^+\right\|_{H^{1/2}(\Gamma_f)} + \|g^-(\alpha)\|_{H^{1/2}(\Gamma_f)} \le 2\left\|g^+\right\|_{H^{1/2}(\Gamma_f)}. \tag{2.4}$$

3. EXISTENCE OF OPTIMAL SHAPE

3.1. Admissible Controls and Domain Convergence

Let the boundary shape to be determined be represented by the graph of the curve

$$\alpha : [a,b] \to [c,d].$$

The domain of the problem, $\Omega = \Omega(\alpha)$, is thus determined by the shape of the unknown boundary Γ_α which we assume to be given by

$$\Gamma_\alpha = \{(x,y) \in [a,b]\times[c,d],\ y = \alpha(x)\},$$

where α is a function to be determined by the optimization process.

As pointed out by Pironneau [7], if Γ_α is the set of all continuous functions, then the optimal shape may not exist. Thus, we need to enforce some restrictions on Γ_α in order for the optimal shape to exist. For our purpose, we define the admissible family of curves as the following uniform Lipschitz continuous functions:

$$A_{\text{ad}} = \{\alpha \in C[a,b];\ c < \alpha < d,\ |\alpha(t_1) - \alpha(t_2)| \le \beta\,|t_1 - t_2|\}.$$

We also need a fixed domain $\hat{\Omega}$ such that $\bigcup_{\alpha\in A_{\text{ad}}} \Omega(\alpha) \subset \hat{\Omega}$. In our setting, we choose $\hat{\Omega}$ to be the region depicted in Figure 1b.

Let $\Omega = \Omega(\alpha)$. We say that $\Omega_n = \Omega(\alpha_n)$ converges to $\Omega = \Omega(\alpha)$ if

$$\|\alpha_n - \alpha\|_\infty := \max_{a\le x\le b} |\alpha_n(x) - \alpha(x)| \to 0, \qquad n \to \infty.$$

The set of domains $\{\Omega(\alpha);\ \alpha \in A_{\text{ad}}\}$ is said to be a set of Lipschitz domains. We will need to following theorem on the extension property of $H^m(\Omega)$.

THEOREM 3.1. *(See [8].) For every uniform Lipschitz domain Ω in R^n and every positive integer m, there exists a linear continuous extension operator*

$$P : H^m(\Omega) \to H^m(R^n), \tag{3.1}$$

such that for each $f \in H^m(\Omega)$,

$$\|Pf\|_{m,R^n} \le C\|f\|_{m,\Omega},$$

where the positive constant C depends only on the Lipschitz constant of the boundary of Ω.

3.2. Weak Formulation of (2.1)

We first convert problem (2.1) into another Helmholtz equation problem with homogeneous boundary conditions. To this end, let u_0 be a function that satisfies

$$\begin{aligned} -\Delta u_0 + u_0 &= 0, \\ u_0|_{\Gamma_f} &= g(\alpha), \\ \left.\frac{\partial u_0}{\partial n}\right|_{\Gamma_\alpha} &= 0, \\ u + \chi \left.\frac{\partial u_0}{\partial n}\right|_{\Gamma_c} &= 0, \\ u + ik \left.\frac{\partial u_0}{\partial n}\right|_{\Gamma_\infty} &= 0. \end{aligned} \tag{3.2}$$

This is a typical elliptic equation problem and it is well known that if $g \in H^{1/2}(\Gamma_f)$, then there exists a unique $u_0 \in H^1(\Omega(\alpha))$ that satisfies the above equations. Furthermore, by (2.4), we have that

$$\|u_0\|_{1,\Omega(\alpha)} \leq \|g(\alpha)\|_{H^{1/2}(\Gamma_f)} \leq C,$$

where C is a constant independent of α. Let $f = f(\alpha) = \Delta u_0 + k^2 u_0$. Then $f \in H^{-1}(\Omega(\alpha))$ and

$$\|f\|_{H^{-1}(\Omega(\alpha))} \leq \left(k^2 + 1\right) \|u_0\|_{1,\Omega(\alpha)} \leq C.$$

Letting $\varphi = u - u_0$, we have that

$$\begin{aligned} \Delta\varphi + k^2\varphi &= f, \\ \varphi|_{\Gamma_f} &= 0, \\ \left.\frac{\partial \varphi}{\partial n}\right|_{\Gamma_\alpha} &= 0, \\ u + \chi \left.\frac{\partial \varphi}{\partial n}\right|_{\Gamma_c} &= 0, \\ iku + \left.\frac{\partial \varphi}{\partial n}\right|_{\Gamma_\infty} &= 0. \end{aligned} \tag{3.3}$$

Define a function space $V_\alpha \subset H^1(\Omega(\alpha))$,

$$V_\alpha := \left\{u \in H^1(\Omega(\alpha));\ u|_{\Gamma_f = 0}\right\}.$$

The variational formulation of (3.3) is to seek $\varphi \in V_\alpha$ such that

$$(\nabla\varphi, \nabla v) - k^2(\varphi, v) + \langle \chi\varphi, v\rangle_{\Gamma_c} + i\langle k\varphi, v\rangle_{\Gamma_\infty} = (f, v), \qquad \text{for } v \in V_\alpha. \tag{3.4}$$

It is well known that with the last boundary condition of (2.2), problem (3.4) has a unique solution [9]. Thus, A_{ad} is a nonempty set.

We now restate our minimization problem using the variation formulation (3.4). First, define the admissibility set V_α of controls and states as follows:

$$V_{\text{ad}} := \{(\alpha, \varphi), \varphi \text{ satisfies (3.4) for } \Omega = \Omega(\alpha)\}.$$

V_{ad} is not empty since A_{ad} is not empty.

We may reformulate our minimization problem as: find $(\alpha, \varphi) \in V_{\text{ad}}$ such that the cost function

$$J(\alpha, \varphi) = A \int_{\Omega(\alpha)} |\varphi + u_0|^2 \, d\Omega + B \int_{\Omega(\alpha)} |\nabla(\varphi + u_0)|^2 \, d\Omega + \lambda \int_a^b (\alpha(x) - \alpha_0(x))^2 \, dx \tag{3.5}$$

is minimized.

To prove our main theorem, we need an important theorem from Fujii [10].

THEOREM 3.2. *Assume that $f(x,u,p)$ is nonnegative and continuous with respect to $(x,u,p) \in R^n \times R \times R^n$ and that it satisfies*

$$f(x,u,p) \to \infty |p| \to \infty, \qquad \textit{for each } (x,u).$$

If $u_n \to u$ weakly in $H^1(\hat{\Omega})$ and $\Omega_n \to \Omega$, then the inequality

$$\int_\Omega f(x,u(x),\nabla u(x))\,dx \le \liminf_{n\to\infty} \int_{\Omega_n} f(x,u_n(x),\nabla u_n(x))\,dx$$

holds.

THEOREM 3.3. *Equation (3.5) has at least one minimizer.*

PROOF. First note that

$$0 < \inf_{(\alpha,\varphi)\in V_{\rm ad}} J(\alpha,\varphi) < \infty.$$

Thus, there exists a minimizing sequence $\{\alpha_n \varphi_n\} \subset V_{\rm ad}$; i.e., (α_n,φ_n) satisfies

$$\lim_{n\to\infty} J(\alpha_n,\varphi_n) = \inf_{(\alpha,\varphi)\in V_{\rm ad}} J(\alpha,\varphi).$$

By Arzela-Ascoli theorem, there exists a subsequence of $\{\alpha_n\}$, which we denote by the same notation, such that

$$\alpha_n \to \alpha^* \text{ uniformly in } [a,b].$$

If $A \neq 0$, then there exists a constant C, independent of n, such that

$$\|\varphi_n\|_{L^2(\Omega_n)} \le 2\|u_0\|^2_{L^2(\Omega_n)} + 2\|\phi_n + u_0\|^2_{L^2(\Omega_n)} \le 2\|u_0\|^2_{L^2(\Omega_n)} + \frac{2}{A}J(\alpha_n,\varphi_n) \le C, \tag{3.6}$$

where C is a constant independent of n. Choosing $v = \varphi_n$ in (3.4), we have that

$$\|\nabla\varphi_n\|^2_{L^2(\Omega_n)} - k^2\|\varphi_n\|^2_{L^2(\Omega_n)} + \langle\chi\varphi_n,\varphi_n\rangle_{\Gamma_c} + i\langle k\varphi_n,\varphi_n\rangle_{\Gamma_\infty} = (f_n,\varphi_n). \tag{3.7}$$

Taking the real parts on both side of the above equation, we obtain

$$\|\nabla\varphi_n\|^2_{L^2(\Omega_n)} - k^2\|\varphi_n\|^2_{L^2(\Omega_n)} + \langle\chi\varphi_n,\varphi_n\rangle_{\Gamma_c} = \mathrm{Re}\,(f_n,\varphi_n)\,.$$

Thus,

$$\|\varphi\|^2_{1,\Omega_n} \le \left(k^2 + C\right)\|\varphi_n\|^2_{L^2(\Omega_n)} + \frac{1}{2}\left(\|\varphi\|^2_{1,\Omega_n} + \|f\|^2_{-1,\Omega_n}\right)$$

or

$$\|\varphi_n\|^2_{1,\Omega_n} \le 2\left(k^2 + C\right)\|\varphi_n\|^2_{L^2(\Omega_n)} + \|f\|^2_{-1,\Omega_n}.$$

The above inequality and (3.6) ensure that there exists a constant C, independent of n, such that

$$\|\varphi_n\|_{1,\Omega_n} \le C.$$

If $A = 0$, then $B \neq 0$ and

$$\|\nabla\varphi_n\|_{L^2(\Omega_n)} \le \frac{1}{B}J(\alpha_n,\varphi_n) \le C,$$

where C is a constant independent of n. So we conclude that there exits a constant C, independent of n such that

$$\|\varphi_n\|_{1,\Omega_n} \le C.$$

Furthermore, due to the uniform extension property, we can choose an extension $\hat{\varphi}_n$ of φ_n to $\hat{\Omega}$ and a positive constant C such that

$$\|\hat{\varphi}_n\|_{1,\hat{\Omega}} \le C\|\varphi_n\|_{1,\Omega_n}.$$

Thus, $\|\hat{\varphi}_n\|_{1,\hat{\Omega}}$ is uniformly bounded in $H^1(\hat{\Omega})$. Consequently, we may extract from the sequence $\{\hat{\varphi}_n\}$ a subsequence, still denoted as $\{\hat{\varphi}_n\}$, such that

$$\hat{\varphi}_n \to \hat{\varphi}, \qquad \text{weakly in } H^1\left(\hat{\Omega}\right).$$

Define $\varphi(\alpha^*) = \hat{\varphi}|_{\Omega(\alpha^*)}$. We shall show that $\varphi(\alpha^*)$ is a solution of (3.4). For this purpose, let us define the function spaces

$$W_n := \left\{\phi \in C^\infty\left(\bar{\Omega}_n\right); \phi = 0 \text{ in the neighborhood of } \Gamma_f\right\}$$

and

$$W := \left\{\phi \in C^\infty\left(\bar{\Omega}\left(\alpha^*\right)\right); \phi = 0 \text{ in the neighborhood of } \Gamma_f\right\}.$$

It is clear that

$$\begin{aligned} V_{\alpha_n} &= H^1_{\Gamma_f}(\Omega_n) = \text{ the closure of } W_n \text{ in } H^1(\Omega_n), \\ V_{\alpha^*} &= H^1_{\Gamma_f}\left(\Omega\left(\alpha^*\right)\right) = \text{ the closure of } W_n \text{ in } H^1\left(\Omega\left(\alpha^*\right)\right). \end{aligned}$$

Let us take $w \in W$ and assume that P is the extension operator on $\Omega(\alpha*)$ defined by (3.1). Let $\hat{w} = Pw$. Then,

$$(\nabla\varphi_n, \nabla\hat{w}) + (\varphi_n, \hat{w}) - k^2\left(\varphi_n, \hat{w}\right) + \langle\chi\varphi_n, \hat{w}\rangle_{\Gamma_c} + i\left\langle k\varphi_n, \hat{w}\right\rangle_{\Gamma_\infty} = (f, \hat{w}). \tag{3.8}$$

To prove that φ is the solution of problem (3.4), we first prove that

$$\int_{\Omega_n} \varphi_n \hat{w}\, d\Omega \to \int_{\Omega(\alpha*)} \varphi w\, d\Omega.$$

Define characteristic functions $\omega = \chi_{\Omega(\alpha^*)}$ and $\omega_n = \chi_{\Omega_n}$. Note that

$$\begin{aligned} \int_{\Omega_n} \varphi_n \hat{w}\, d\Omega - \int_{\Omega(\alpha*)} \varphi w\, d\Omega &= \int_{\hat{\Omega}} \hat{\varphi}_n \omega_n \hat{w}\, d\Omega - \int_{\hat{\Omega}} \hat{\varphi}\omega\hat{w}\, d\Omega \\ &= \int_{\hat{\Omega}} (\omega_n - \omega)\hat{\varphi}_n \hat{w}\, d\Omega + \int_{\hat{\Omega}} \left(\hat{\varphi}_n - \hat{\varphi}\right)\omega\hat{w}\, d\Omega. \end{aligned}$$

Since $\|\alpha_n - \alpha^*\|_\infty \to 0$, $\omega_n \to \omega$ strongly in $L^2(\hat{\Omega})$. From this and the fact that $\hat{\varphi}_n \to \hat{\varphi}$ weakly in $H^1(\hat{\Omega})$, we have that

$$\left|\int_{\hat{\Omega}} (\omega_n - \omega)\hat{\varphi}_n \hat{w}\, d\Omega\right| \le C\|\omega_n - \omega\|_{L^2(\hat{\Omega})}\|\hat{\varphi}_n\|_{L^2(\hat{\Omega})} \le C\|\omega - \omega_n\|_{L^2(\hat{\Omega})} \to 0$$

and

$$\int_{\hat{\Omega}} \left(\hat{\varphi}_n - \hat{\varphi}\right)\omega\hat{w}\, d\Omega \to 0.$$

Thus,

$$\int_{\Omega_n} \varphi_n \hat{w}\, d\Omega \to \int_{\Omega(\alpha*)} \varphi w\, d\Omega.$$

Similarly, we can show that

$$\int_{\Omega_n} \nabla\varphi_n \nabla\hat{w}\, d\Omega \to \int_{\Omega(\alpha*)} \nabla\varphi\nabla w\, d\Omega.$$

Next we show that

$$\langle\varphi_n, w\rangle_{\Gamma_c} \to \langle\varphi, w\rangle_{\Gamma_c}.$$

By the trace theorem, we have that

$$\|\varphi_n - \varphi\|_{H^{1/2}(\Gamma_c)} \le \|\varphi_n - \varphi\|_{1,\Omega_n} \le C,$$

where C is a constant independent of n. Thus, we may extract a subsequence from φ_n, still denoted with the same notation, such that $\varphi_n \to \varphi$ weakly in $H^{1/2}(\Gamma_c)$ which implies that $\varphi_n \to \varphi$ strongly in $L^2(\Gamma_c)$. Thus,

$$|\langle\varphi_n, w\rangle_{\Gamma_c} - \langle\varphi, w\rangle_{\Gamma_c}| = |\langle\varphi_n - \varphi, w\rangle_{\Gamma_c}| \le \|\varphi_n - \varphi\|_{L^2(\Gamma_c)}\|w\|_{L^2(\Gamma_c)} \to 0.$$

Similarly, we can show that

$$\langle\chi\varphi_n, w\rangle_{\Gamma_c} \to \langle\chi\varphi, w\rangle_{\Gamma_c}.$$

Letting $n \to \infty$ in (3.8), we conclude that $\varphi = \varphi(\alpha^*)$ satisfies (3.4). Thus, $(\alpha^*, \varphi(\alpha^*)) \in V_{\text{ad}}$. The proof of the theorem is concluded by the fact that

$$J(\alpha^*, \varphi(\alpha^*)) \le \liminf_{n\to\infty} J(\alpha_n, \varphi_n) = \lim_{n\to\infty} J(\alpha_n, \varphi_n) = \inf_{(\alpha,\varphi)\in V_{\text{ad}}} J(\alpha, \varphi).$$

4. SPECTRAL ELEMENT MODEL

A discretization of equation (3.4) is obtained using the spectral element method [11]. Let $N \ge 1$ be a specified integer, and denote by $\mathcal{Q}_N$ the set of complex polynomials in the variables (x, y) such that the degree of any $q \in \mathcal{Q}_N$ in each variable does not exceed N. The domain $\Omega(\alpha)$ is partitioned into E generalized nonoverlapping quadrilateral elements which can have curved boundaries such that $\Omega(\alpha) = \bigcup_{e=1}^{E} \Omega_e$. Given the shape of the boundary $\partial\Omega$ in terms of control points on the fan casing, these elements are generated using a simple mesh generation strategy. The trial solution is sought in the space $V_\alpha^N \subset V_\alpha$ of complex-valued functions whose restrictions to an element are polynomials in $\mathcal{Q}_N$, i.e., $V_\alpha^N = \{\phi \in V_\alpha : \phi|_{\Omega_e} \in \mathcal{Q}_N\}$. In order to construct a spectral approximation based on Chebyshev polynomials, let us consider the Gauss-Chebyshev-Lobatto points in $[-1, 1]$ defined by $\xi_i = \eta_i = -\cos(\pi i/N)$, and define the master domain $\Omega_M = [-1, 1]^2$. A one-to-one transformation from Ω_M onto any arbitrary element Ω_e, given by $(x, r) = \mathcal{M}(\xi, \eta)$, is supposed to exist. Such a mapping can be constructed by transfinite interpolation [12] from a description of the edges of the element Ω_e. A suitable basis for the trial space can then be constructed from the images of the Lagrange interpolation polynomials based on the points (ξ_i, η_j) through the transformation $\mathcal{M}(\xi, \eta)$. On each element, the trial solution takes the form

$$\phi(x, y) = \sum_{i,j=0}^{N} \phi_{ij} h_i(\xi(x, y)) h_j(\eta(x, y)), \tag{4.1}$$

where $\phi_{ij} = \phi(\mathcal{M}(\xi_i, \eta_j))$. The $(N+1)^2$ test functions used for the Galerkin projection are then $\psi_{kl}(x, y) = h_k(\xi(x, y))h_l(\eta(x, y))$, where k, l also vary from 0 to N.

The Lagrange interpolants on the master element Ω_M can be conveniently expressed [11] in terms of the Chebyshev polynomials $T_p(x) = \cos(p \arccos(x))$, due to the particular position of the points ξ_i

$$h_i(\xi) = \prod_{p\ne i}^{N} \left(\frac{\xi - \xi_p}{\xi_i - \xi_p}\right) = \frac{2}{Nc_i} \sum_{p=0}^{N} \frac{T_p(\xi_i)T_p(\xi)}{c_p}, \tag{4.2}$$

with $c_i = 2$ if $i = 0$ or $i = N$ and $c_i = 1$, otherwise. For the test function ψ_{kl} then, the volume integrals in (x, y) space can be recast as volume integrals over the master element in the form

$$I_{kl} = \int_{-1}^{1}\int_{-1}^{1} \mathcal{L}_\phi \phi(\xi,\eta)\mathcal{L}_\psi \psi_{kl}(\xi,\eta) f(\xi,\eta)\, d\xi\, d\eta, \tag{4.3}$$

where the operator $\mathcal{L} \in \{1, \frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}\}$. The function $f(\xi,\eta)$ contains a product of terms related to the Jacobian of the transformation, $J = |\frac{\partial(x,y)}{\partial(\xi,\eta)}|$, and the corresponding metric terms that occur through the change of variables. The evaluation of the integral is performed by replacing this function with its spectral interpolant,

$$f(x,r) = \sum_{m,n=0}^{N} f(\mathcal{M}(\xi_m,\eta_n))h_m(\xi)h_n(\eta), \tag{4.4}$$

such that the integral becomes

$$I_{kl} = \sum_{i,j=0}^{N} S_{kl}^{ij}\phi_{ij}, \tag{4.5}$$

where

$$S_{kl}^{ij} = \sum_{m,n=0}^{N} f_{mn} \int_{-1}^{1} \mathcal{L}_\phi^i h_i(\xi)\mathcal{L}_\psi^k h_k(\xi)h_m(\xi)\,d\xi \int_{-1}^{1} \mathcal{L}_\phi^j h_j(\eta)\mathcal{L}_\psi^l h_l(\eta)h_n(\eta)\,d\eta. \tag{4.6}$$

The operators acting on the Lagrange interpolants are such that, for example, $\mathcal{L}_\phi^i = \frac{\partial}{\partial\xi}$ if $\mathcal{L}_\phi = \frac{\partial}{\partial\xi}$ and unity otherwise. The integrals on the right-hand side can now be evaluated by replacing the interpolants by their expression (4.2) and directly computing the integrals of products of Chebyshev polynomials.

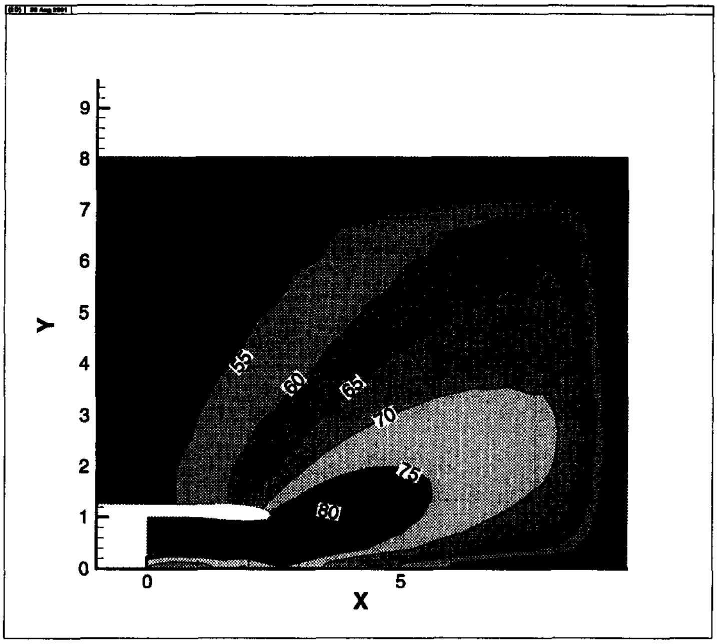
In order to limit the size of the domain, the radiation boundary condition is imposed using a damping layer within which the amplitude of the wave decays exponentially with distance [3]. The resulting system of linear equations with complex coefficients is solved using a direct solver. The sensitivities of the objective function with respect to the control points on the fan casing were computed by a discrete adjoint of the resulting computer code, and a steepest descent method used them to drive the objective function towards a minimum.

5. NUMERICAL RESULTS

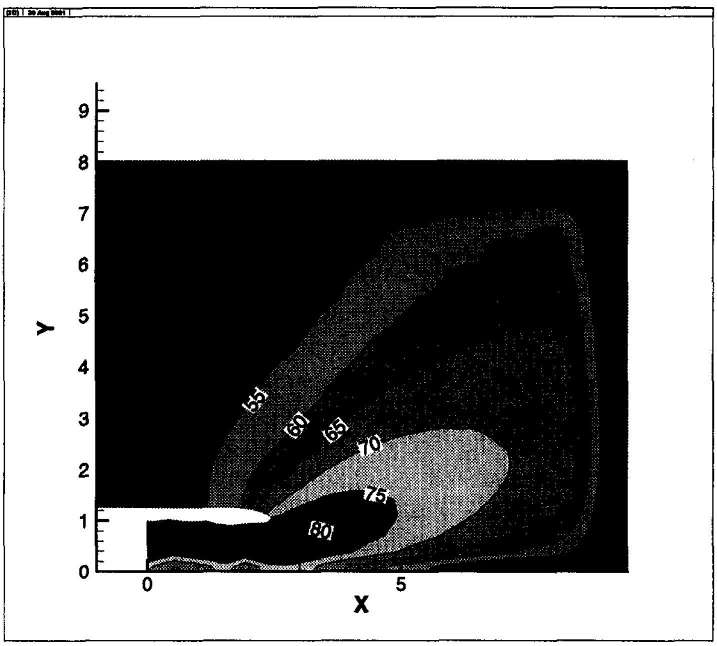
In order to verify the feasibility of the approach, several numerical experiments have been performed aiming at optimization of fan inlet shapes under the incidence of various acoustic modes. To find the minimizer of the cost function, we use the optimization subroutine 789 for bounded domain. The discretized adjoint equation method is used to calculate the gradient of the cost function. Figure 2 presents the contours of such an optimization performed starting from a generic inlet shape, for a wave number $k = 8$. For the first azimuthal order, propagation of the first two radial modes $(1,0)$ and $(1,1)$ with equal amplitude and phase has been modeled. The constraint on the objective function was imposed by restraining the deformation of the initial boundary. Optimization subroutine "778" [13] is used to find the minimizer of the cost function. A decrease by about 25% of the objective function is achieved. As can be noticed from the value of the contours, this leads to a valuable decrease of between one and two decibels in the sound pressure levels at points situated in the main lobe region. By industrial standard, such a result is significant and clearly useful, taking into account the small deformation of the shape of the nacelle.

6. CONCLUSIONS

The shape optimization problem for noise radiation from ducted fans has been studied here. Our results show that it may present one viable alternative for far field noise reduction. The extension of this work to the case with mean flow represents a natural extension of these results which we hope to address in the near future.



(a)



(b)

Figure 2. Sound pressure level (SPL$=10\log_{10}(u\bar{u})+100$) contours for an initial fan inlet shape (a) and the optimized one (b).

REFERENCES

1. W. Eversman, I.D. Roy and H.D. Meyer, Improved finite element modeling of the turbofan engine inlet radiation problem, NASA Report, Contract NAS3-25952 (1993).
2. D. Stanescu, D. Ait-Ali-Yahia, W.G. Habashi and M.P. Robichaud, Galerkin spectral element method for fan tone radiation computation, AIAA Paper 2000-1912 (2000).

3. D. Stanescu, D. Ait-Ali-Yahia, W.G. Habashi and M. Robichaud, Multidomain spectral computations of sound radiation from ducted fans, *AIAA Journal* **37**, 296–302 (1999).
4. D. Chenais, On the existence of a solution in a domain identification problem, *J. Math. Anal. Appl.* **52**, 189–219 (1975).
5. M. Gunzburger and H. Lee, Existence of an optimal solution of a shape control problem for the stationary Navier-Stokes equations, *SIAM J. Control Optim.* **36**, 859–909 (1998).
6. D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer-Verlag, (1991).
7. O. Pironneau, *Optimal Shape Design for Elliptic Systems*, Springer-Verlag, (1984).
8. J. Marti, *Introduction to Soblev Spaces and Finite Element Solution of Elliptic Boundary Value Problems*, Academic Press, London, (1986).
9. F. Ihlenburg, *Finite Element Analysis of Acoustic Scattering*, Springer, (1998).
10. N. Fujii, Low semicontinuity in domain optimization problems, *J. Opt. Appl.* **59** (1988).
11. A.T. Patera, A spectral element method for fluid dynamics: Laminar flow in a channel expansion, *J. of Computational Physics* **54** (3), 468–488 (1984).
12. J.S. Hesthaven, A stable penalty method for the compressible Navier-Stokes equations. III. Multi dimensional domain secomposition achemes, *SIAM J. on Scientific Computing* **20** (1), 62–93 (1999).
13. R.H. Byrd, P. Lu, J. Nocedal and C. Zhu, A limited memory algorithm for bound constrained optimization, *SIAM J. Scient. Comp.* **16**, 1190–1208 (1995).