Potential Anal (2008) 28:301–319
DOI 10.1007/s11118-008-9078-4

# Finite Element Method and Discontinuous Galerkin Method for Stochastic Scattering Problem of Helmholtz Type in $\mathbb{R}^d$ ($d = 2, 3$)

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Received: 29 May 2007 / Accepted: 21 February 2008 /
Published online: 18 March 2008


**Abstract** In this paper, we address the finite element method and discontinuous Galerkin method for the stochastic scattering problem of Helmholtz type in $\mathbb{R}^d$ ($d = 2, 3$). Convergence analysis and error estimates are presented for the numerical solutions. The effects of the noises on the accuracy of the approximations are illustrated. Results of the numerical experiments are provided to examine our theoretical results.

**Keywords** Finite element method · Discontinuous Galerkin method · Stochastic Helmholtz equation

**Mathematics Subject Classifications (2000)** 65C30 · 65N30

## 1 Introduction

Many physical and engineering phenomena are modeled by partial differential equations which often contain some levels of uncertainty. The complexity for the

The first author is supported by NSF under grand number 0609918 and AFOSR under grant numbers FA9550-06-1-0234 and FA9550-07-1-0154, the second author is supported by NSFC(10671082, 10626026, 10471054), and the third author is supported by NNSF (No. 10701039 of China) and 985 program of Jilin University.

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solution of these so-called stochastic partial differential equations (SPDE) is that SPDEs are able to better capture the behavior of interesting phenomena; it also means that the corresponding numerical analysis of the model will require new tools to model the systems, produce the solutions, and analyze the information stored within the solutions. In the last decade, many researchers have studied different SPDEs and various numerical methods and approximation schemes for SPDEs have also been developed, analyzed, and tested [1, 3, 6, 9, 11–17, 19, 23]. In [3, 13], the analysis based on the traditional finite element method was successfully used on SPDEs with random coefficients, using the tensor product between the deterministic and random variable spaces. Numerical methods for SPDEs with white noise and Brownian motion added to the forcing term have also been studied [3, 6, 9, 10].

In this paper, we study the following stochastic scattering problem of Helmholtz type driven by an additive white noise (here we propose and study a perfectly matched layer technique for solving exterior Helmholtz problems with perfectly conducting boundary):

$$\begin{cases} \Delta u(x) + k^2 u(x) = -f(x) - \dot{W}(x), & x \in \mathbb{R}^d \setminus \Omega, \\ \dfrac{\partial u(x)}{\partial n} = g(x), & x \in \partial\Omega, \\ \lim\limits_{r\to\infty} r^{(d-1)/2}\left(\dfrac{\partial u}{\partial r} - iku\right) = 0, & r = |x|. \end{cases} \tag{1}$$

Here $\Omega$ is a bounded convex domain with smooth boundary of $\mathbb{R}^d (d = 2, 3)$, $g \in H^{-\frac{1}{2}}(\partial\Omega)$ is determined by the incoming wave, and $n$ is the unit outer normal to $\partial\Omega$. We assume the source term has compact support belonging in $\Omega_1 \setminus \Omega (\Omega \subset \Omega_1)$, i.e. $supp(f + \dot{W}) \subset \Omega_1 \setminus \Omega$ . Here, we assume throughout the paper that the wave number $k$ is positive and bounded away from zero, i.e., $k \geq k_0 > 0$. The existence and uniqueness of the weak solutions for Eq. 1 have been established in [5] by converting the problem into the Hammerstein integral equation using the Green function.

The difficulty in the error analysis of finite element methods and general numerical approximations for a SPDE is the lack of regularity of its solution. For instance, as shown in [3], the required regularity conditions are not satisfied for the stochastic elliptic problem for the standard error estimates of finite element methods. For one dimension case, if $\dot{W}$ corresponds to the Brownian white noise, Allen, Novosel, and Zhang have shown that the regularity estimates are usually very weak, and lead to very low order error estimates [3]. On the other hand, if the noise is more regular, Du and Zhang have proved that it is possible to get higher order of error estimates for the numerical solution. To the best of our knowledge, there exist few work in the literature which study the finite element method and discontinuous Galerkin method for the stochastic scattering problem of Helmholtz type in $\mathbb{R}^d$ $(d = 2, 3)$.

The goal of this work is to derive error estimates for numerical solutions of Eq. 1 using finite element approximations and discontinuous Galerkin method. The key to the error estimates is the Lipschitz type regularity properties of the Green functions in $L^2$ norm. In two dimensional case, such a regularity result was obtained in [6] for the elliptic case. In this paper we derive an improved estimate for the regularity of the Green function both two and three dimensional case. As a result we obtain error estimates for the finite element approximations in 3-D case which are comparable to finite difference error estimates.

The paper is organized as follows. In Section 2, we study the approximation of Eq. 1 using discretized white noises with PML technique, and establish the error estimate in $L^2$-norm. In Section 3, we study finite element method and discontinuous Galerkin method of the stochastic Helmholtz equation with discretized white noises, and obtain the $L^2$ error estimates between numerical solution and the exact solution of Eq. 1. Finally, in Section 4, we present numerical simulations using the finite element method and discontinuous Galerkin method constructed in the Section 3.

## 2 Discretized White Noises and PML Technique

### 2.1 The Approximation Problem Using Discretized White Noises

In this subsection, we first introduce the approximate problem of Eq. 1 by replacing the white noise $\dot{W}$ with its piecewise constant approximation $\dot{W}^s$. Then we establish the regularity of the solution of the approximate problem and its error estimates. For the simplicity of presentation, we assume that $\Omega$ is a convex polygonal domain and the source term in Eq. 1 has compact support belonging in $\Omega_1 \setminus \Omega (\Omega \subset \Omega_1)$, i.e. $supp(f + \dot{W}) \subset \Omega_1 \setminus \Omega$. We remark that the results in this paper can be easily extended to solve the scattering problems with other boundary conditions such as Dirichlet or the impedance boundary condition on $\partial\Omega$.

Suppose we are given a triangulation $\mathcal{T}_{1h} = \bigcup_{i=1}^{n_1} K_i$ on $\Omega_1 \setminus \Omega$, consisting of simplices and the elements $K_i \in \mathcal{T}_{1h}$ may have one curved edge aligned with $\partial\Omega$. Let $h_i = \text{diam} K_i$, $h = \max_{1 \le i \le n_1} h_i$ and $\rho_i$ be the radius of the largest ball inscribed in $K_i$. We say an element $K_i \in \mathcal{T}_{1h}$ is $\sigma_0$-*shape regular* if

$$h_i/\rho_i \le \sigma_0, \quad 1 \le i \le n_1,$$

and $\mathcal{T}_{1h}$ is $\sigma_0$-shape regular if all its elements are $\sigma_0$-shape regular. Moreover, we say $\mathcal{T}_{1h}$ is quasi-uniform if it is shape regular and satisfies

$$h \le \sigma_1 h_i, \quad 1 \le i \le n_1,$$

with $\sigma_0$ and $\sigma_1$ fixed positive constants. Write

$$\xi_i = \frac{1}{\sqrt{|K_i|}} \int_{K_i} 1 dW(x)$$

for $1 \le i \le n_1$, where $|K_i|$ denotes the area of $K_i$. It is well-known that $\{\xi_i\}_{i=1}^{n_1}$ is a family of independent identically distributed normal random variables with mean 0 and variance 1 (see [22]). Then the piecewise constant approximation to $\dot{W}(x)$ is given by

$$\dot{W}^{s(h)}(x) = \sum_{i=1}^{n_1} |K_i|^{-\frac{1}{2}} \xi_i \chi_i(x), \tag{2}$$

where $\chi_i$ is the characteristic function of $K_i$. It is easy to see that $\dot{W}^{s(h)} \in L^2(\Omega)$. However, we have the following estimate which shows that $\|\dot{W}^{s(h)}\|$ is unbounded as $h \to 0$(c.f. [6]).

**Lemma 1** *There exist positive constants $C_1$ and $C_2$ independent of $h$ such that*

$$C_1 h^{-2} \leq E\left(\|\dot{W}^{s(h)}\|^2_{L^2(\Omega_1\setminus\Omega)}\right) \leq C_2 h^{-2}. \tag{3}$$

Replacing $\dot{W}$ by $\dot{W}^{s(h)}$ in Eq. 1, we have the following "simple" problem:

$$\begin{cases} \Delta u^{s(h)}(x) + k^2 u^{s(h)}(x) = -f(x) - \dot{W}^{s(h)}(x), & x \in \mathbb{R}^d \setminus \Omega, \\ \dfrac{\partial u^{s(h)}(x)}{\partial n} = -g(x), & x \in \partial\Omega, \\ \lim\limits_{r\to\infty} r^{(d-1)/2}\left(\dfrac{\partial u^{s(h)}}{\partial r} - iku^{s(h)}\right) = 0, & r = |x|. \end{cases} \tag{4}$$

Let $\Omega_1$ be contained in the interior of the circle or ball $B_R = \{x \in \mathbb{R}^d : |x| < R\}$. We start by introducing an equivalent variational formulation of Eq. 4 in the bounded domain $\Omega_R = B_R \setminus \overline{\Omega}$. Let $T : H^{\frac{1}{2}}(\partial B_R) \to H^{-\frac{1}{2}}(\partial B_R)$ be the Dirichlet-to-Neumann operator(c.f. [8]), and let $a_2 : H^1(\Omega_R) \times H^1(\Omega_R) \to \mathbb{C}$ be the sesquilinear form:

$$a_2(u, v) = \int_{\Omega_R} \left(\nabla u \cdot \nabla \overline{v} - k^2 u \overline{v}\right) dx - \langle Tu, v\rangle_{\partial B_R}, \tag{5}$$

where $\langle\cdot,\cdot\rangle_{\partial B_R}$ stands for the inner product on $L^2(\partial B_R)$ or the duality pairing between $H^{\frac{1}{2}}(\partial B_R)$ and $H^{-\frac{1}{2}}(\partial B_R)$, and similar notation applies for $\langle\cdot,\cdot\rangle_{\partial\Omega}$, $\langle\cdot,\cdot\rangle_{\partial B_\rho}$. The scattering problem Eq. 4 is equivalent to the following weak formulation: Find $u^{s(h)} \in H^1(\Omega_R)$ such that

$$a_2\left(u^{s(h)}, v\right) = \langle g, v\rangle_{\partial\Omega} + \int_{\Omega_R} F^{s(h)} v, \quad \forall\, v \in H^1(\Omega_R), \tag{6}$$

where $F^{s(h)} = f + \dot{W}^{s(h)}$. The existence of a unique solution of the variational problem Eq. 6 is known (c.f. [8]).

Next we estimate the error between the weak solution $u$ of Eq. 1 and its approximation $u^{s(h)}$. Recall that $u$ and $u^{s(h)}$ are the unique solutions of the following Hammerstein integral equations, respectively (c.f. [4]):

$$u = \int_{\Omega_1\setminus\Omega} G(x, y) f(y)dy + \int_{\Omega_1\setminus\Omega} G(x, y)dW(y) + \int_{\partial\Omega} \frac{\partial G(x, y)}{\partial \nu} g(y)ds(y), \tag{7}$$

$$u^s = \int_{\Omega_1\setminus\Omega} G(x, y) f(y)dy + \int_{\Omega_1\setminus\Omega} G(x, y)dW^{s(h)}(y) + \int_{\partial\Omega} \frac{\partial G(x, y)}{\partial \nu} g(y)ds(y), \tag{8}$$

where $G(x, y)$ is the Green function of the Helmholtz equation. It is well-known that

$$G(x, y) = \begin{cases} G_2(x, y) = \frac{1}{2\pi}\log\frac{1}{|x-y|} + V_2(x, y), & when\ d = 2, \\ G_3(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|} + V_3(x, y), & when\ d = 3. \end{cases} \tag{9}$$

where $V_i(x, y), i = 2, 3$ are Lipschitz continuous functions of $x$ and $y$ (c.f. [8]).

The following lemma regarding the regularity of the Green function $G$ defined in Eq. 9 will play an important role in the estimate.

**Lemma 2**

(a) *For $d = 2$, there exists a positive number $C$ independent of $\alpha \in (0, 1)$ s.t.*

$$\int_{\Omega_1\setminus\Omega} |G_2(x, y) - G_2(x, z)|^2 dx \le C\alpha^{-1}|y - z|^{2-\alpha}, \qquad \forall\, y, z \in \Omega. \tag{10}$$

(b) *For $d = 3$, there exists a positive number $C$ independent of $\gamma = \min\{3-\beta, \beta\}$ s.t.*

$$\int_{\Omega_1\setminus\Omega} |G_3(x, y) - G_3(x, z)|^\beta dx \le C|y - z|^\gamma, \qquad \forall\, y, z \in \Omega,\ \beta \in (1, 3). \tag{11}$$

*Proof* We only show Eq. 11 holds for the singular part of $G_3$ here, the proof of Eq. 10 is the same as in Lemma 2 in [6]. Let $\xi = (y + z)/2, r = |y - z|$, $B_{r_1}(x)$ denote the ball with center $x$ and radius $r_1$ and assume $\Omega_1 \setminus \Omega \subset B_{r_2}(0)$. Obvious, we have

$$\int_{\Omega_1\setminus\Omega}\left|\frac{1}{|x-y|} - \frac{1}{|x-z|}\right|^\beta dx = I + II + III + IV$$

$$:= \int_{B_{\frac{r}{4}}(y)} + \int_{B_{\frac{r}{4}}(z)} + \int_{B_{5r}(\xi)\setminus B_{\frac{r}{4}}(y)\cup B_{\frac{r}{4}}(z)} + \int_{(\Omega_1\setminus\Omega)\setminus B_{5r}(\xi)}\left|\frac{1}{|x-y|} - \frac{1}{|x-z|}\right|^\beta dx.$$

Next, we estimate the above four parts item by item,

$$\begin{aligned} I &:= \int_{B_{\frac{r}{4}}(y)}\left|\frac{1}{|x-y|} - \frac{1}{|x-z|}\right|^\beta dx = \int_{B_{\frac{r}{4}}(y)}\frac{||x-z| - |x-y||^\beta}{|x-y|^\beta|x-z|^\beta}dx \\ &\le 2^\beta\int_{B_{\frac{r}{4}}(y)}\frac{|y-z|^\beta}{|x-y|^\beta r^\beta}dx = 2^\beta\int_{B_{\frac{r}{4}}(y)}\frac{dx}{|x-y|^\beta} \qquad \left(|x-z| \ge \frac{r}{2}\right) \\ &\le C\int_0^{\frac{r}{4}}\frac{s^2}{s^\beta}ds \le Cr^{3-\beta}, \end{aligned}$$

similarly, we have $II \le Cr^{3-\beta}$ and

$$\begin{aligned} III &:= \int_{B_{5r}(\xi)\setminus B_{\frac{r}{4}}(y)\cup B_{\frac{r}{4}}(z)}\left|\frac{1}{|x-y|} - \frac{1}{|x-z|}\right|^\beta dx \\ &\le \frac{C}{r^\beta}\int_{B_{5r}(\xi)\setminus B_{\frac{r}{4}}(y)\cup B_{\frac{r}{4}}(z)} 1dx \le Cr^{3-\beta}. \end{aligned}$$

Note that $|\frac{|x-\xi|}{|x-y|}-1| \leq \frac{|\xi-y|}{|x-y|} \leq \frac{r/2}{5r-(r/2)} = \frac{1}{9}$, we have $\frac{8}{9}|x-y| \leq |x-\xi| \leq \frac{10}{9}|x-y|$, and similarly $\frac{8}{9}|x-z| \leq |x-\xi| \leq \frac{10}{9}|x-z|$, then

$$IV := \int_{(\Omega_1\setminus\Omega)\setminus B_{5r}(\xi)} \left|\frac{1}{|x-y|} - \frac{1}{|x-z|}\right|^{\beta} dx \leq \int_{(\Omega_1\setminus\Omega)\setminus B_{5r}(\xi)} \frac{r^{\beta}}{|x-y|^{\beta}|x-z|^{\beta}} dx$$

$$\leq Cr^{\beta} \int_{(\Omega_1\setminus\Omega)\setminus B_{5r}(\xi)} \frac{1}{|x-\xi|^{2\beta}} dx \leq Cr^{\beta} \int_{5r}^{r_2} \frac{s^2}{s^{2\beta}} ds \leq Cr^{\beta}\left(r^{3-2\beta} + r_2^{3-2\beta}\right)$$

$$\leq C\left(r^{\beta} + r^{3-\beta}\right).$$

Combining all the above inequalities, we obtain the desired estimate Eq. 11 by setting $\gamma = \min\{3-\beta, \beta\}$. □

*Remark 1* Setting $\alpha = \frac{1}{|\log|y-z||}$ and $\beta = 2$ in Eqs. 10 and 11 respectively, we obtain

$$\int_{\Omega_1\setminus\Omega} |G_2(x,y) - G_2(x,z)|^2 dx \leq C|y-z|^2 \log|y-z|, \qquad \forall\, y, z \in \Omega_1 \setminus \Omega, \tag{12}$$

$$\int_{\Omega_1\setminus\Omega} |G_3(x,y) - G_3(x,z)|^2 dx \leq C|y-z|, \qquad \forall\, y, z \in \Omega_1 \setminus \Omega. \tag{13}$$

Now we are in a position to establish the error estimate between $u$ and $u^s$.

**Theorem 1** *If $u$ and $u^{s(h)}$ are the solution of Eqs. 1 and 4 respectively, then for any bounded domain $\Omega_M = B_M \setminus \overline{\Omega}$ $(\Omega_1 \subset B_M)$, we have*

$$E\left(\|u - u^{s(h)}\|^2_{L^2(\Omega_M)}\right) = \begin{cases} Ch^2|\log h|, & d = 2, \\ Ch, & d = 3, \end{cases} \tag{14}$$

*where $C$ is a positive constant independent of $u$ and $h$.*

*Proof* Subtracting Eq. 7 from 8, we obtain

$$u - u^{s(h)} = \int_{\Omega_1\setminus\Omega} G(x,y)dW(y) - \int_{\Omega_1\setminus\Omega} G(x,y)dW^{s(h)}(y). \tag{15}$$

Using the Ito isometry we have that

$$
\begin{aligned}
&E\left(\|u-u^{s(h)}\|^2_{L^2(\Omega_M)}\right)\\
&\quad = E\left(\int_{\Omega_1\setminus\Omega}\left[\int_{\Omega_1\setminus\Omega} G(x,y)dW(y) - \int_{\Omega_1\setminus\Omega} G(x,y)dW^s(y)\right]^2 dx\right)\\
&\quad = E\left(\int_{\Omega_1\setminus\Omega}\left[\sum_{i=1}^{n_1}\int_{K_i} G(x,y)dW(y) - |K_i|^{-1}\sum_{i=1}^{n_1}\int_{K_i} G(x,z)dz\int_{K_i} 1dW(y)\right]^2 dx\right)\\
&\quad = E\left(\int_{\Omega_1\setminus\Omega}\left[\sum_{i=1}^{n_1}\int_{K_i}\int_{K_i}|K_i|^{-1}(G(x,y)-G(x,z))dzdW(y)\right]^2 dx\right)\\
&\quad = \int_{\Omega_1\setminus\Omega}\left(\sum_{i=1}^{n_1}\int_{K_i}\left[|K_i|^{-1}\int_{K_i}(G(x,y)-G(x,z))dz\right]^2 dy\right)dx. \qquad (\mathit{Ito\ isometry})
\end{aligned}
$$

From the Hölder inequality, we obtain

$$
\begin{aligned}
E\left(\|u-u^{s(h)}\|^2_{L^2(\Omega_M)}\right) &\le \int_{\Omega}\left(\sum_{i=1}^{n_1}|K_i|^{-1}\int_{K_i}\int_{K_i}(G(x,y)-G(x,z))^2dzdy\right)dx\\
&= \sum_{i=1}^{n_1}|K_i|^{-1}\int_{K_i}\int_{K_i}\int_{\Omega}(G(x,y)-G(x,z))^2dxdzdy.
\end{aligned}
$$

Then the results Eq. 14 follows from the above inequality, Remark 1 and Eq. 15. □

### 2.2 PML Technique for Approximation Problem

Now we turn to the introduction of the absorbing PML layer following [7]. We surround the domain $\Omega_R$ with a PML layer $\Omega_P = \{x \in \mathbb{R}^d : R < |x| < \rho\}$ (see Fig. 1). The specially designed model medium in the PML layer should basically be so chosen that either the wave never reaches its external boundary or the amplitude of the reflected wave is so small that it does not essentially contaminate the solution in $\Omega_R$. Throughout the paper we assume $\rho \le CR$ for some generic fixed constant $C > 0$.

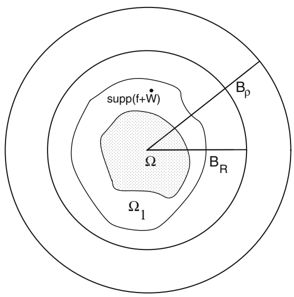



**Fig. 1** Setting of the scattering problem with the PML layer

Let $\alpha(r) = 1 + i\sigma(r)$ be the fictitious medium property which satisfies

$$\sigma \in C(\mathbb{R}), \quad \sigma \geq 0, \quad and \quad \sigma = 0 \quad for \quad r \leq R.$$

Denote by $\widetilde{r}$ the complex radius defined by

$$\widetilde{r} = \widetilde{r}(r) = \begin{cases} r, & if\ r \leq R, \\ \int_0^r \alpha(s)ds = r\beta(r), & if\ R \leq r \leq \rho. \end{cases} \tag{16}$$

Using the relation

$$\frac{\widetilde{r}}{r} = \beta(r) \quad and \quad \frac{\partial \widetilde{r}}{\partial r} = \alpha(r),$$

the complexified Helmholtz-like equation can be rewritten as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\beta}{\alpha} r \frac{\partial w}{\partial r}\right) + \frac{\alpha}{\beta}\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + k^2 \alpha\beta w = -f(\beta r) - \dot{W}^{s(h)}(\beta r). \tag{17}$$

Letting $A$ be the matrix which satisfies, in polar coordinates,

$$\nabla \cdot (A\nabla) = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\beta}{\alpha} r \frac{\partial}{\partial r}\right) + \frac{\alpha}{\beta}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

then the PML equation is given by

$$\nabla \cdot (A\nabla w) + k^2 \alpha\beta w = -f(\beta r) - \dot{W}^{s(h)}(\beta r), \quad in\ \Omega_P.$$

The PML solution $u_P^{s(h)}$ in $\Omega_\rho = B_\rho \setminus \overline{\Omega}$ is defined as the solution of the following system

$$\begin{cases} \nabla \cdot \left(A\nabla u_P^{s(h)}\right) + \alpha\beta k^2 u_P^{s(h)} = -f - \dot{W}^{s(h)}, & in\ \Omega_\rho, \\ \dfrac{\partial u_P^{s(h)}}{\partial n} = -g \quad on\ \partial\Omega, \quad u_P^{s(h)} = 0 \quad on\ \partial B_\rho. \end{cases} \tag{18}$$

This problem can be reformulated in the bounded domain $\Omega_R$ by imposing the boundary condition

$$\frac{\partial u_P^{s(h)}}{\partial n}\Big|_{\partial B_R} = T_P u_P^{s(h)},$$

where the operator $T_P : H^{\frac{1}{2}}(\partial B_R) \to H^{-\frac{1}{2}}(\partial B_R)$ is defined as follows: Given $q_1 \in H^{\frac{1}{2}}(\partial B_R)$,

$$T_P q_1 = \frac{\partial \zeta}{\partial n}|_{\partial B_R},$$

where $\zeta \in H^1(\Omega_P)$ satisfies

$$\begin{cases} \nabla \cdot (A\nabla \zeta) + \alpha\beta k^2 \zeta = -f - \dot{W}^{s(h)}, & in\ \Omega_P, \\ \zeta = q_1 \quad on\ \partial B_R, \quad \zeta = 0 \quad on\ \partial B_\rho. \end{cases} \tag{19}$$

Based on the operator $T_P$, we introduce the sesquilinear form $a_P : H^1(\Omega_R) \times H^1(\Omega_R) \to \mathbb{C}$ by:

$$a_P(u, v) = \int_{\Omega_R} \left(A\nabla u \cdot \nabla \overline{v} - k^2\alpha\beta u\overline{v}\right) dx - \langle T_P u, v\rangle_{\partial B_R}. \tag{20}$$

Then the weak formulation for Eq. 20 is: Find $u_P^{s(h)} \in H^1(\Omega_R)$ such that

$$a_P(u_P^{s(h)}, v) = \langle g, v\rangle_{\partial\Omega} + \int_{\Omega_R} F^{s(h)} v, \quad \forall\, v \in H^1(\Omega_R). \tag{21}$$

The existence and uniqueness of the solutions of the PML problems Eq. 21 is known (c.f. [7]).

In order to give the convergence results about the PML problem, we first make the following two assumptions:

**(H1)** In practical applications, the fictitious medium property $\sigma$ is usually taken as power functions:

$$\sigma(r) = \sigma_0 \left(\frac{r - R}{\rho - R}\right)^m, \quad R \le r \le \rho, \quad \textit{for some constant } \sigma_0 > 0 \textit{ and integer } m \ge 1,$$

which is rather mild in the practical application of the PML techniques.

**(H2)** There exists a unique solution to the following Dirichlet PML problem in the layer $\Omega_P$:

$$\begin{cases} \nabla \cdot (A\nabla w) + \alpha\beta k^2 w = -f - \dot{W}^{s(h)}, & in\ \Omega_P, \\ w = 0 \quad on\ \partial B_R, \quad w = q_2 & on\ \partial B_\rho, \end{cases} \tag{22}$$

where $q_2 \in H^{\frac{1}{2}}(\partial B_\rho)$.

**Lemma 3** (c.f. [7]) *Let* **(H1)-(H2)** *be satisfied. Then for sufficiently large* $\sigma_0 > 0$, *the PML problem Eq. 21 has a unique solution* $u_P^{s(h)} \in H^1(\Omega_\rho)$. *Moreover, we have the following estimate*

$$\|u^s - u_P^{s(h)}\|_{H^1(\Omega_R)} \le C(1 + kR)^2 e^{-k\, Im(\tilde{\rho})\left(1 - \frac{R^2}{|\tilde{\rho}|}\right)^{1/2}} \|u_P^{s(h)}\|_{H^{\frac{1}{2}}(\partial B_R)}, \tag{23}$$

*where* $\widetilde{\rho} = \widetilde{\rho}(\rho)$ *given by Eq. 16.*

## 3 Finite Element Method and Discontinuous Galerkin Method

In this section, we consider the finite element and discontinuous Galerkin approximations of the variational problem Eq. 4 for low wave-numbers as well as mid-frequency case and establish their error estimates.

3.1 Finite Element Methods

We introduce the finite element approximations of the PML problems Eq. 18. From now on we assume $g \in L^2(\partial\Omega)$. Let $b_P : H^1(\Omega_\rho) \times H^1(\Omega_\rho) \to \mathbb{C}$ be the sesquilinear form given by:

$$b_P(u, v) = \int_{\Omega_\rho} \left(A\nabla u \cdot \nabla \overline{v} - k^2 \alpha\beta u \overline{v}\right) dx. \tag{24}$$

Denote by $V(\Omega_\rho) = \{v \in H^1(\Omega_\rho) : v = 0 \ on \ \partial B_\rho\}$. Then the weak formulation of Eq. 18 is: Find $u_P^{s(h)} \in V(\Omega_\rho)$ such that

$$b_P\left(u_P^{s(h)}, v\right) = \int_{\partial\Omega} g\overline{v} ds + \int_{\Omega_\rho} F^{s(h)} v, \quad \forall\, v \in V(\Omega_\rho). \tag{25}$$

Suppose we are given regular triangulation $\mathcal{T}_{2h} = \bigcup_{i=n_1+1}^{n_2} K_i$ on $\Omega_\rho \setminus \Omega_1$(the elements $K_i \in \mathcal{T}_{2h}$ may have one curved edge align with $\partial B_\rho$), consisting of simplices, and then $\mathcal{T}_h = \mathcal{T}_{1h} \cup \mathcal{T}_{2h} = \bigcup_{i=1}^{n_2} K_i$ is a regular triangulation of the domain $\Omega_\rho$, i.e. we use the same mesh size of finite element approximation as white noises discretization.

Let $V^h \subset H^1(\Omega_\rho)$ be the conforming linear finite element space over $\Omega_\rho$, and $V_0^h = \{v_h \in V^h : v_h = 0 \ on \ \partial B_\rho\}$. The finite element approximation to the PML problem Eq. 18 reads as follows: Find $u_{P,h}^{s(h)} \in V_0^h$ such that

$$b_P\left(u_{P,h}^{s(h)}, v\right) = \int_{\partial\Omega} g\overline{v} ds + \int_{\Omega_\rho^h} F^{s(h)} v, \quad \forall\, v \in V_0^h. \tag{26}$$

**Lemma 4** (c.f. [18]) *Let $D$ be a bounded domain with smooth boundary. Then, there exist positive constants $C$ depending only on $D$ and the angles of the triangulation so that, under the assumption $hk^2 \lesssim 1$, the finite element solution $u_{P,h}^{s(h)}$ satisfies*

$$\left|\left\| u_P^{s(h)} - u_{P,h}^{s(h)} \right\|\right|_D \le C \inf_{\chi \in V_0^h} \left|\left\| u_P^{s(h)} - \chi \right\|\right|_D \le Chk \left( \left\| F^{s(h)} \right\|_{L^2(D)} + \left\| g \right\|_{H^{1/2}(\partial D)} \right),$$

*where $C$ only depends on $k_0$.*

The following theorem is the main result of this section.

**Theorem 2** *Assume $\Omega_R$ is a bounded domain with smooth boundary, and $u$ and $u_{P,h}^{s(h)}$ are the solution of Eqs. 1 and 26 respectively. Assume* **(H1)-(H2)** *are satisfied , the mesh satisfies $hk^2 \lesssim 1$ and $k^2$ is not an exterior eigenvalue. Then for sufficiently large $\sigma_0 > 0$, we have*

$$E\left(\|u - u_{P,h}^{s(h)}\|^2_{L^2(\Omega_R)}\right) = Ce^{-k\, Im(\tilde{\rho})\left(1 - \frac{R^2}{|\tilde{\rho}|}\right)^{1/2}} + \begin{cases} Ch^2|\log h| + Ch^2k^2, & d = 2, \\ Ch + Ch^2k^2, & d = 3, \end{cases} \tag{27}$$

*where $C$ is a positive constant independent of $h$.*

*Proof* By the triangulation inequality and Aubin-Nitsche technique ([20]), we have

$$\left\|u-u_{P,h}^{s(h)}\right\|_{L^2(\Omega_R)}^2 \le 2\left(\left\|u-u^{s(h)}\right\|_{L^2(\Omega_R)}^2+\left\|u^{s(h)}-u_P^{s(h)}\right\|_{L^2(\Omega_R)}^2+\left\|u_P^{s(h)}-u_{P,h}^{s(h)}\right\|_{L^2(\Omega_R)}^2\right)$$
$$\le C\Bigg(\left\|u-u^{s(h)}\right\|_{L^2(\Omega_R)}^2+\left\|u^{s(h)}-u_P^{s(h)}\right\|_{H^1(\Omega_R)}^2$$
$$+Ck^2h^2\left\|u_P^{s(h)}-u_{P,h}^{s(h)}\right\|_{H^1(\Omega_R)}^2\Bigg).$$

Using Theorem 1, Lemma 3, Lemma 4 for the last three terms, we obtain the conclusion. □

### 3.2 Discontinuous Galerkin Method

The standard finite element method provides a quasi-optimal numerical method for elliptic boundary value problems in the sense that the accuracy of the numerical solution differs only by a constant $C$ from the best approximation from the finite element space. While this property guarantees good performances of computations at any mesh resolution for the Laplace operator, it can not be preserved for the Helmholtz equation. The reason is that last term in Eq. 27 increases with the wavenumber $k$. This phenomenon is well-known as a pollution effect. It is due to numerical dispersion errors and finite element method are able to cope with large wave numbers only if the mesh resolution is also increased suitably (under the constraint $hk^2 \lesssim 1$). In order to avoid the pollution effect, numerous discretization techniques have been developed. They include the weak element method for the Helmholtz equation, the Galerkin/least-squares method, the quasi-stabilized finite element method, the partition of unity method, the residual-free bubbles for the Helmholtz equation, the ultra-weak variational method, the least squares method, and recently a discontinuous Galerkin method has been introduced in numerical simulations by Alvarez et al.(c.f. [2]).

Here we shall analyze discontinuous Galerkin (DG) discretizations of the stochastic scattering problem of Helmholtz type with Dirichlet boundary condition $u = g$ on $\partial\Omega$. Let $\mathcal{T}_h = \bigcup_{i=1}^{n_2} K_i$ be specified in Section 3.1 with set of edges $\mathcal{E}_h$. We define new function spaces:

$$H(\Omega_\rho) = \left\{\phi \in L^2(\Omega_\rho) \mid \phi \in H^1(K_i) \ \ and \ \ \Delta\phi \in L^2(K_i) \ \ for \ \ i = 1, ...n_2\right\},$$
$$U_{DG} = \left\{\phi \in H(\Omega_\rho) \mid \phi = g \ \ on \ \ \partial\Omega,\ \phi = 0 \ \ on \ \ \partial B_\rho\right\},$$
$$V_{DG} = \left\{v \in H(\Omega_\rho) \mid v = 0 \ \ on \ \ \partial\Omega,\ v = 0 \ \ on \ \ \partial B_\rho\right\},$$

and redefine the PML problem Eq. 18 as: find $u_P^{s(h)} \in H(\Omega_\rho)$ that satisfies:

$$\nabla\cdot\left(A\nabla u_{i,P}^{s(h)}\right)+\alpha\beta k^2 u_{i,P}^{s(h)} = -f_i-\dot{W}_i^{s(h)}, \quad in \quad K_i,$$
$$u_{i,P}^{s(h)}(x) = g_i(x), \quad on \quad \partial K_i\cap\partial\Omega, \quad u_P^{s(h)} = 0 \quad on\ \partial K_i\cap\partial B_\rho,$$

with interface conditions

$$u_i^{s(h)} - u_{i'}^{s(h)} = 0, \quad \left(\nabla u_i^{s(h)} - \nabla u_{i'}^{s(h)}\right) \cdot \mathbf{n}_i = 0 \quad a.e. \quad K_i \cap K_{i'},$$

where $u_{i,P}^{s(h)}$ is the restriction of the function $u_P^{s(h)}$ to the subdomain $K_i, i = 1, \cdots, n_2$.

The family of discontinuous methods introduced here relies on a variational formulation to the above problem including jump terms across the element edges, as follows: find $u_P^{s(h)} \in U_{DG}$ that satisfies:

$$a_2\left(u_P^{s(h)}, v\right) = a_G\left(u_P^{s(h)}, v\right) + a_{DG}\left(u_P^{s(h)}, v\right) = \left(-f - \dot{W}^{s(h)}, v\right), \qquad \forall\, v \in V_{DG},$$

with

$$a_G(u, v) = \sum_{i=1}^{n_2} \int_{K_i} \left(A\nabla u_i \cdot \nabla v_i - k^2 \alpha\beta u_i v_i\right) dx,$$

$$a_{DG}(u, v) = \sum_{i=1}^{n_2} \sum_{i'>i}^{n_2} \int_{K_i \cap K_{i'}} \left[ -\frac{A}{2}(\nabla u_i + \nabla u_{i'}) \cdot \mathbf{n}_i (v_i - v_{i'}) \right.$$
$$\left. + \frac{\beta_{ii'}}{h_{ii'}}(u_i - u_{i'})(v_i - v_{i'}) + \frac{\lambda_{ii'}}{2}(u_i - u_{i'}) A(\nabla v_i + \nabla v_{i'}) \cdot \mathbf{n}_i \right] ds,$$

where $h_{ii'} = \min\{h_i, h_{i'}\}$, $\beta_{ii'}$ and $\lambda_{ii'}$ are functions to be determined aiming at reducing the pollution effects compared to standard finite element formulations.

We define DG norm and weighted DG norm as

$$\|v\|_{DG}^2 = \sum_{i=1}^{n_2} \int_{K_i} \left(|A\nabla v_i|^2 + \alpha\beta v_i^2\right) dx + \sum_{i=1}^{n_2} \sum_{i'>i}^{n_2} \int_{K_i \cap K_{i'}} \frac{1}{h_{ii'}} (v_i - v_{i'})^2 ds,$$
$$|||v|||_{DG}^2 = \|v\|_{DG}^2 + k^2 \|v\|_{0,\Omega_\rho}^2,$$

and the discontinuous finite dimension spaces as

$$U_{DG,h} = \left\{u \in L^2(\Omega_\rho) \mid u_i \in P^1(K_i) \;\; and \;\; u = g_h \;\; on \;\; \partial\Omega, \;\; u = 0 \;\; on \;\; \partial B_\rho\right\},$$
$$V_{DG,h} = \left\{v \in L^2(\Omega_\rho) \mid v_i \in P^1(K_i) \;\; and \;\; v = 0 \;\; on \;\; \partial\Omega, \;\; v = 0 \;\; on \;\; \partial B_\rho\right\},$$

where $P^1$ is the piecewise polynomial of degree one, then the corresponding finite element approximation yields: find $u_{P,h}^{s(h)} \in U_{DG,h}$ that satisfies

$$a_2\left(u_{P,h}^{s(h)}, v\right) = a_G\left(u_{P,h}^{s(h)}, v\right) + a_{DG}\left(u_{P,h}^{s(h)}, v\right) = \left(-f - \dot{W}^{s(h)}, v\right), \quad \forall\, v \in V^h = V_{DG,h}, \tag{28}$$

The residual of the DG formulation is defined as follows:

$$\mathcal{R}_h\left(u_{P,h}^{s(h)}, v\right) = \left(f + \dot{W}^{s(h)}, v\right) + a_G\left(u_{P,h}^{s(h)}, v\right) + a_{DG}\left(u_{P,h}^{s(h)}, v\right), \quad \forall\, v \in V^h = V_{DG,h}.$$

The analysis will be carried out by adapting Perugia's argument [21] beginning with following abstract error bound.

**Lemma 5** *Let $u_P^{s(h)}$ and $u_{P,h}^{s(h)}$ be the solution to Eqs. 18 and 28 respectively, we have*

$$\left\|\left| u_P^{s(h)} - u_{P,h}^{s(h)} \right|\right\|_{DG} \le C \left( \inf_{\chi \in V^h} \left\|\left| u_P^{s(h)} - \chi \right|\right\|_{DG} + k^2 \sup_{0 \neq \chi \in V^h} \frac{\left( u_P^{s(h)} - u_{P,h}^{s(h)}, \chi \right)}{\| \chi \|_{0,\Omega_\rho}} + \sup_{0 \neq \chi \in V^h} \frac{\mathcal{R}_h \left( u_P^{s(h)}, \chi \right)}{\| \chi \|_{DG}} \right), \tag{29}$$

*with $C > 0$ independent of $h$ and $k$.*

Next, we consider the error estimates of the three terms on the right hand side of Eq. 29 separately. The first term is just the best approximation error. By the standard Aubin–Nitsche technique, we can get following estimate for the second term.

**Lemma 6** *Let $u_P^{s(h)}$ be the solution to Eq. 18 belongs to $H^2(\Omega_\rho)$, we have*

$$\left( u_P^{s(h)} - u_{P,h}^{s(h)}, \chi \right) \le Ch \left( (1+kh) \left\|\left| u_P^{s(h)} - u_{P,h}^{s(h)} \right|\right\|_{DG} + \left( \sum_{i=1}^{n_2} h_i^2 \left\| u_P^{s(h)} \right\|_{2,K_i}^2 \right)^{1/2} \right) \| \chi \|_{0,\Omega_\rho}, \tag{30}$$

*with $C > 0$ independent of $h$ and $k$.*

For the third term, by the DG method for the residual term [21], we have following lemma.

**Lemma 7** *Let $u_P^{s(h)}$ be the solution to Eq. 18 belongs to $H^2(\Omega_\rho)$, then*

$$\mathcal{R}_h \left( u_P^{s(h)}, \chi \right) \le C \left( \sum_{i=1}^{n_2} h_i^2 \left\| u_P^{s(h)} \right\|_{2,K_i}^2 \right)^{1/2} \| h^{-1/2} [[\chi]]_N \|_{0,\mathcal{E}_h}^2 \quad \forall\, \chi \in V_h. \tag{31}$$

*with $C > 0$ independent of $h$ and $k$.*

We can state our main result of this section as follows:

**Theorem 3** *Let $u$ and $u_h^{s(h)}$ be the solution of Eqs. 1 and 28 respectively. If the mesh satisfies $hk^2(1+hk) \gtrsim 1$ and $h$ small enough, we have*

$$E\left( \left\| u - u_{P,h}^{s(h)} \right\|_{L^2(\Omega_R)}^2 \right) = Ce^{-k\, Im(\tilde{\rho}) \left( 1 - \frac{R^2}{|\tilde{\rho}|} \right)^{1/2}} + \begin{cases} Ch^2 |\log h| + Ch^2 k^2, & d = 2, \\ Ch + Ch^2 k^2, & d = 3, \end{cases} \tag{32}$$

*with $C > 0$ independent of $h$ and $k$.*

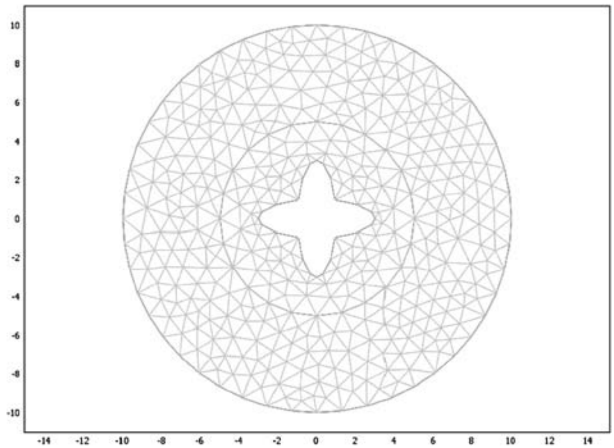



**Fig. 2** The partition of the 2D scattering problem

*Proof* Insert the estimate Eq. 30 into Eq. 29 and subtract $Ch\|u_P^{s(h)} - u_{P,h}^{s(h)}\|_{DG}$ from both sides of Eq. 29. Using the best approximation error and Lemma 7, we obtain

$$\left\|\left\| u_P^{s(h)} - u_{P,h}^{s(h)} \right\|\right\|_{DG} \le C \left( \sum_{i=1}^{n_2} h_i^2 \|u^{s(h)}\|_{2,K_i}^2 \right)^{1/2},$$

provided that $hk^2(1+hk) \gtrsim 1$. The above inequality, Theorem 1, Lemma 3 and Aubin-Nitsche technique together imply the conclusion of the Theorem. □

## 4 Numerical Results for Some Model Equations

In this section, we present numerical examples to demonstrate our theoretical results in the previous section. We will consider finite element method and discontinuous Galerkin method for stochastic scattering problem of Helmholtz type.

The normal random variables for $\dot{W}^{s(h)}$ shall be simulated by using the random number generator of femlab. Theoretically, the number of samples $M$ should be chosen so that the error generated by the Monte Carlo method is of the same magnitude as the errors generated by the finite element approximation. For the linear problem, $E(u_{P,h}^{s(h)})$ is the finite element approximation of the deterministic solution,

**Table 1** Finite element method for Example 1 when $k=1$: Test 1

| $h$ | $e_1$ | Rate | $E\left(\|u_{P,h}^{s(h)}\|_{L^2(\Omega_R)}^2\right)$ | $e_2$ | Rate |
|---|---|---|---|---|---|
| 1/8 | 3.81e-2 | | 20.56264 | 3.93e-2 | |
| 1/16 | 1.12e-2 | 1.76 | 20.59089 | 1.11e-2 | 1.83 |
| 1/32 | 3.21e-3 | 1.81 | 20.59901 | 2.94e-3 | 1.91 |

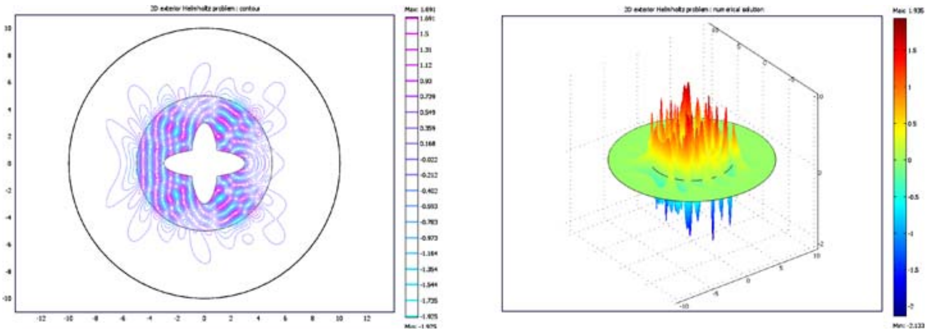


**Fig. 3** The numerical solution of $E(u)$ by finite element method when $N = 64$: the *left* is contour and the *right* is the surface

and we shall evaluate $E(u_{P,h}^{s(h)})$ by using the Monte Carlo method to examine

$$e_1(h) = \left\| E(u) - E\left(u_{P,h}^{s(h)}\right) \right\|^2$$

to see if we have used enough samples. We also employ the following type of errors

$$e_2(h) = \left| E\left(\|u\|^2\right) - E\left(\left\|u_{P,h}^{s(h)}\right\|^2\right)\right|$$

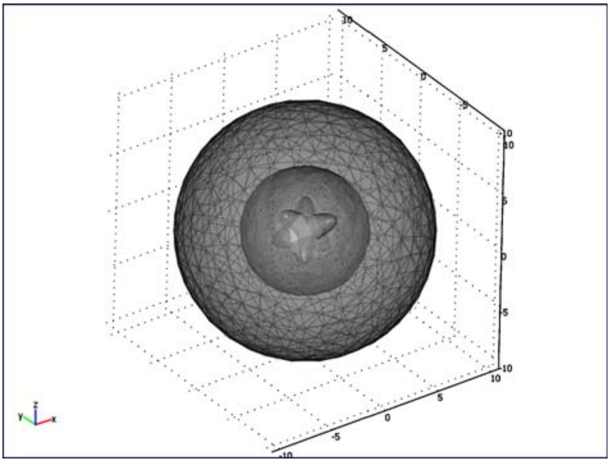
to check error estimates for finite element method and discontinuous Galerkin method, respectively. Obviously these two errors can be controlled by the error $E\left(\|u - u_{P,h}^{s(h)}\|^2\right)^{\frac{1}{2}}$, but are not equivalent to it. Nevertheless we believe that they provide good indications about how the error $E\left(\|u - u_{P,h}^{s(h)}\|^2\right)^{\frac{1}{2}}$ itself behaves. Notice that it is impossible to evaluate $|E\left(\|u - u_{P,h}^{s(h)}\|\right)|$ since it is impossible to obtain an explicit expression for $u$.

In the next two examples, we test the convergence rates for the stochastic scattering problem of Helmholtz type. According to the discussion in Section 2.2, we choose the PML medium property as the power function and thus we need only to specify the thickness $\rho - R$ of the layer and the medium parameter $\sigma_0$. Recall from

**Table 2** Discontinuous Galerkin method for Example 1 when $k = 10$: Test 2

| $h$ | $e_1$ | Rate | $E\left(\|u_{P,h}^{s(h)}\|_{L^2(\Omega_R)}^2\right)$ | $e_2$ | Rate |
|---|---|---|---|---|---|
| 1/8 | 3.12e-2 | | 20.56724 | 3.47e-2 | |
| 1/16 | 9.02e-2 | 1.79 | 20.59225 | 9.69e-2 | 1.84 |
| 1/32 | 2.52e-3 | 1.84 | 20.59931 | 2.63e-3 | 1.88 |

**Fig. 4** The partition of the 3D scattering problem



Theorem 2, in our implementation we choose $\rho$ and $\sigma_0$ such that the exponentially decaying factor:

$$e^{-k\,Im(\tilde{\rho})\left(1-\frac{R^2}{|\tilde{\rho}|}\right)^{1/2}} \leq 10^{-6},$$

which makes the PML error negligible compared with the finite element or discontinuous Galerkin discretization errors.

*Example 1* Let the scatter $\Omega$ be union of two ellipses, the support of the source be $\Omega_1(=B_5(0))\setminus\Omega$, the PML region be $\Omega_P=B_{10}(0)\setminus B_5(0)$, and the partition of the scattering problem is shown as Fig. 2. We test the performance of finite element method for solving problem Eq. 1 with $f=\frac{1}{(x^2+y^2)^{\frac{2}{3}}}+\frac{1}{(x^2+y^2)^{\frac{1}{3}}}$, and the scattering of the plane wave $u_i=e^{ikx}$ from a perfectly conducting metal (i.e. $g=0$). We use $h=\frac{1}{64}$ as the finest mesh and obtain numerically

$$E\left(\|u_{fine}\|^2_{L^2(\Omega_R)}\right)=20.601942.$$

The computational results of finite element method for Eq. 1 with $k=1$ are displayed in Table 1.The the second and third columns of the tables show that the rate of convergence for $E(u^{s(h)}_{P,h})$ is about order 2, which implies that our sample sizes are good enough to ensure the accuracy of the Monte Carlo method. The contour and surface figures of numerical solution for $E(u)(N=64)$ are present by Fig. 3. The computational results of discontinuous Galerkin method for Eq. 1 with $k=10$ are

**Table 3** Finite element method for Example 2 when $k=1$: Test 3

| $h$ | $e_1$ | Rate | $E\left(\|u^{s(h)}_h\|^2_{L^2(\Omega_R)}\right)$ | $e_2$ | Rate |
|---|---|---|---|---|---|
| 1/4 | 4.23e-2 | | 55.22873 | 5.61e-2 | |
| 1/8 | 1.29e-2 | 1.71 | 55.25147 | 3.33e-2 | 0.75 |
| 1/16 | 3.76e-3 | 1.78 | 55.26581 | 1.91e-2 | 0.81 |

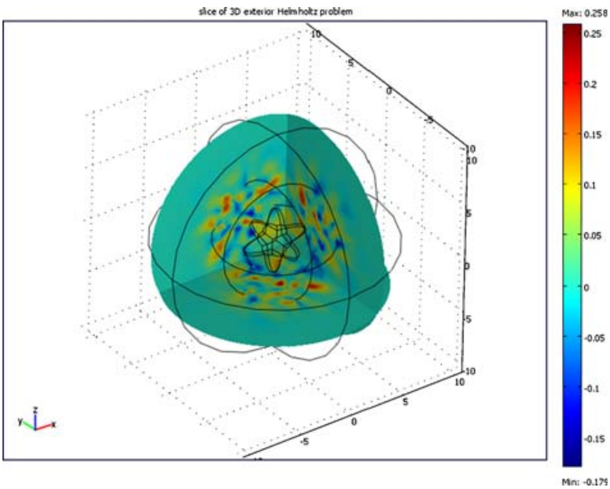



**Fig. 5** The numerical solution of $E(u)$ by finite element method when $N = 32$

displayed in Table 2. We point out that the numerical results of using finite element method for $k = 10$ does not work well as discontinuous Galerkin method.

*Example 2* Let the scatter $\Omega$ be the union of three ellipsoids, the support of the source be $\Omega_1(= B_5(0)) \setminus \Omega$, the PML region be $\Omega_P = B_{10}(0) \setminus B_5(0)$, and the partition of the scattering problem is shown as Fig. 4. We test the performance of finite element method for solving problem Eq. 1 with $f = \frac{9}{x^2+y^2+z^2} - \frac{6}{(x^2+y^2+z^2)^2}$, and the scattering of the plane wave $u_i = e^{ikx}$ from a perfectly conducting metal (i.e. $g = 0$). We use $h = \frac{1}{32}$ as the finest mesh and obtain numerically

$$E\left(\|u_{fine}\|^2_{L^2(\Omega_R)}\right) = 55.28483.$$

The computational results of finite element method for Eq. 1 with $k = 1$ are displayed in Table 3. The the second and third columns of the tables show that the rate of convergence for $E(u^{s(h)}_{P,h})$ is about order 2, which implies that our sample sizes are good enough to ensure the accuracy of the Monte Carlo method. The slice figure of numerical solution for $E(u)(N = 32)$ are present by Fig. 5. The computational results of discontinuous Galerkin method for Eq. 1 with $k = 10$ are displayed in Table 4.

**Table 4** Discontinuous Galerkin method for Example 2 when $k = 10$: Test 4

| $h$ | $e_1$ | Rate | $E\left(\|u^{s(h)}_h\|^2_{L^2(\Omega_R)}\right)$ | $e_2$ | Rate |
|---|---|---|---|---|---|
| 1/4 | 4.87e-2 | | 55.22443 | 6.04e-2 | |
| 1/8 | 1.53e-2 | 1.67 | 55.24791 | 3.69e-2 | 0.71 |
| 1/16 | 4.46e-3 | 1.72 | 55.26333 | 2.15e-2 | 0.78 |

## 5 Conclusion

In this paper, we develop finite element method and discontinuous Galerkin method for stochastic scattering problem of Helmholtz type driven by additive white noises in $\mathbb{R}^d$ ($d = 2, 3$). More importantly, we allow the domain to be any bounded domain with smooth boundary (or a bounded convex domain), not just a rectangle, which is the main advantage of the finite element over other methods such as finite difference methods and spectral finite element methods. Results of the numerical experiments are provided to valid our theoretical results.

**Acknowledgements** The authors wish to thank the anonymous referee for many constructive comments which lead to some essential improvements of the paper.

## References

1. Allen, E.J., Novosel, S.J., Zhang, Z.: Difference approximation of some linear stochastic partial differential equations. Stochastics Stochastics Rep. **64**, 117–142 (1998)
2. Alvarez, G.B., Loula, A.F.D., Dutra, E.G., Rochinha, F.A.: A discontinuous finite element formulation for Helmholtz equation. Comput. Methods Appl. Mech. Engrg. **195**(33–36), 4018–4035 (2006)
3. Babuska, I., Tempone, R., Zouraris, G.: Galerkin fnite element approximations of stochastic elliptic partial differential equations. SIAM J. Numer. Anal. **42**, 800–825 (2004)
4. Brenner, S., Scott, L.R.: The Mathematical Theory of Finite Element Methods. Springer, Berlin Heidelberg New York (1994)
5. Buckdahn, R., Pardoux, E.: Monotonicity methods for white noise driven quasi-linear PDEs. Progr. Probab. **22**, 219–233 (1990)
6. Cao, Y.Z., Yang, H.T., Yin, L.: Finite element methods for semilinear elliptic stochastic partial differential equations. Numer. Math. **106**(2), 181–198 (2007)
7. Chen, Z.M., Liu, X.Z.: An adaptive perfectly matched layer technique for time-harmonic scattering problems. SIAM J. Numer. Anal. **43**(2), 645–671 (2005)
8. Colton, D., Kress, R.: Integral Equation Methods in Scattering Theory. Wiley, New York (1983)
9. Du, Q., Zhang, T.Y.: Numerical approximation of some linear stochastic partial differential equations driven by special additive noise. SIAM J. Numer. Anal. **4**, 1421–1445 (2002)
10. Gyöngy, I.: Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise II. Potential Anal. **11**, 1–37 (1999)
11. Hausenblas, E.: Error analysis for approximation of stochastic differential equations driven by Poisson random measures. SIAM J. Numer. Anal. **40**, 87–113 (2002)
12. Johnson, C.: Numerical Solution of Partial Differential Equations by the Finite Element Method. Cambridge University Press, Cambridge (1994)
13. Keese, A., Matthies, H.: Parallel computation of stochastic groundwater flow. In: Wolf, D., Munster, G., Kremer, M. (eds.) NIC Symposium 2004, Proceedings, NIC series, vol. 20, pp. 399–408. John von Neumann Institute for Computing, Julich (2003)
14. Kloeden, P., Platen, E.: Numerical Solution of Stochastic Differential Equations. Springer, New York (1992)
15. Lin, Y.P., Zhang, K., Zou, J.: Studies on some perfectly matched layers for one-dimensional time-dependent systems. Adv. Comput. Math. (2008, in press)
16. Martínez, T., Sanz-Solé, M.: A lattice scheme for stochastic partial differential equations of elliptic type in dimension $d \geq 4$. Appl. Math. Optim. **54**(3), 343–368 (2006)
17. McLean, W.: Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, Cambridge (2000)
18. Melenk, J.M.: On generalized finite element methods. Ph.D. thesis, University of Maryland at College Park (http://www.anum.tuwien.ac.at/~melenk/) (1995)
19. Milstein, G.N., Platen, E., Schurz, H.: Balanced implicit methods for stochastic systems. SIAM J. Numer. Anal. **35**, 1010–1019 (1998)
20. Mitchell, A.R., Griffiths, D.F.: The Finite Difference Method in Partial Differential Equations. Wiley, Chichester (1980)

21. Perugia, I., Schötzau, D.: An *hp*-analysis of the local discontinuous Galerkin method for diffusion problems. J. Sci. Comput. **17**, 561–571 (2002)
22. Walsh, J.B.: An Introduction to Stochastic Partial Differential Equations. Lecture Notes in Mathematics, vol. 1180, pp. 265–439. Springer, Berlin Heidelberg New York (1986)
23. Zhang, K., Zhang, R., Yin, Y.G., Yu, S.: Domain decomposition methods for linear and semi-linear elliptic stochastic partial differential equations. Appl. Math. Comput. (2008 in press)