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Finite element methods for semilinear elliptic stochastic partial differential equations

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Abstract We study finite element methods for semilinear stochastic partial differential equations. Error estimates are established. Numerical examples are also presented to examine our theoretical results.

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1 Introduction

In recent years, it has become clear that many phenomena, both in nature and in engineering, which are commonly described by systems of deterministic partial differential equations, may be more fully modeled by systems of stochastic partial differential equations (SPDEs) instead. However, the complexity of the SPDE model is carried over to the solutions themselves, which are no longer simple functions, but instead stochastic processes. This complexity of the solutions is the reason that SPDEs are able to more fully capture the behavior of

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interesting phenomena; it also means that the corresponding numerical analysis of the model will require new tools to model the systems, produce the solutions, and analyze the information stored within the solutions.

Indeed the numerical analysis of SPDEs has become a highly active research area in the past few years. SPDEs derived from fluid flow and other engineering fields have been studied using Wiener chaos expansions in [4,7,14,17,21,23]. In [3,15], the analysis based on the traditional finite element method was successfully used on SPDEs with random coefficients, using the tensor product between the deterministic and random variable spaces. Numerical methods for SPDEs with white noise and Brownian motion added to the forcing terms have also been developed, analyzed, and tested by numerous authors [2,9,11,13,19,20,24,25],

In this paper, we study finite element methods for the following boundary value problem of a semilinear stochastic elliptic partial differential equation driven by an additive white noise:

$$\begin{cases} -\Delta u(x) + f(u(x)) = g(x) + \dot{W}(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open set of R^2, $\dot{W}(x)$ is a white noise, $g \in L^2(\Omega)$ and f is a continuous function on $\overline{\Omega}$ satisfying certain regularity conditions given in Sect. 2. The existence and uniqueness of the weak solution for (1.1) have been established in [6] by converting the problem into the Hammerstein integral equation using the Green's function. The integral equation is also used as a tool to derive the error estimates of the numerical approximations for problem (1.1) (see [2,9,11]).

The difficulty in the error analysis of finite element methods and general numerical approximations for a SPDE is the lack of regularity of its solution. For instance, as shown in [2], the required regularity conditions are not satisfied for problem (1.1) for the standard error estimates of finite element methods. To overcome this difficulty, Allen, Novosel, and Zhang [2] and Du and Zhang [9] consider replacing $\dot{W}$ with its piecewise constant approximation. Then the solution of the resulting SPDE has the desired regularity for the error estimates of finite element methods for $\Omega = (a, b)$.

The main challenge to carrying out an error analysis of the above finite element approach for the SPDE (1.1) in higher dimensional spaces is the lack of regularity of the Green's function for the Laplacian operator. When the domain $\Omega = (a, b)$, the Green's function for the Laplacian operator is a simple Lipschitz-continuous function; but this is not the case when Ω is a domain in higher dimensional spaces. In this work we provide a Lipschitz-type regularity estimate for the Green's function of the Laplacian operator in the L^2 norm. This allows us to obtain an error estimate for the approximation of (1.1) with discretized white noises. Notice that we allow Ω to be any convex domain in R^2, not just a rectangle. This extension to general domains is one of the major advantages of finite element methods over other methods such as finite difference methods and spectral methods.

Nonlinearity in (1.1) is another challenge for the error analysis of finite element methods. Because of the lack of regularity of the exact solution, we must use the L^2-norm to estimate the errors of the approximate solutions. For linear problems, this can be done easily using a duality argument. Here we shall use the Galerkin projection operator to resolve the difficulty.

The paper is organized as follows. In Sect. 2, we study the approximation of (1.1) using discretized white noises. We shall establish an estimate for the approximate solution in H^2-norm and its error estimate in L^2-norm. In Sect. 3, we study a finite element method of the SPDE with discretized white noises and obtain the L^2 error estimate between the finite element solution and the exact solution of (1.1). Finally, in Sect. 4, we present numerical simulation results using the finite element method constructed in Sect. 3.

To conclude the introduction, we introduce the notations that will be used throughout the paper. For an integer m, we use $H^m(\Omega)$ to denote the usual Sobelev space whose norm is denoted by $\|\cdot\|_m$. When $m = 0$, $H^0(\Omega)$ shall be denoted by $L^2(\Omega)$, the space of square integrable functions on Ω. Its inner product and norm are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. We also use $H_0^1(\Omega)$ for the subsapce of $H^1(\Omega)$ whose elements vanish on the boundary of Ω.

2 The approximate problem

In this section, we first introduce the approximate problem for (1.1) by replacing the white noise $\dot{W}$ with its piecewise constant approximation $\dot{W}_h$. Then we establish the regularity of the solution of the approximate problem and its error estimates. Without loss of generality, we shall assume that $f(0) = 0$. Otherwise, we just replace f by $f - f(0)$ and g by $g - f(0)$. For the simplicity of presentation we will also assume that Ω is a convex polygonal domain.

Let $\{\mathcal{T}_h\}$ be a family of triangulations of $\overline{\Omega}$ (see [5] for the requirements on $\{\mathcal{T}_h\}$), where $h \in (0,1)$ is the meshsize. We assume the family is quasiuniform, i.e., there exist positive constants ρ_1 and ρ_2 such that

$$\rho_1 h \le R_T^{\mathrm{inr}} < R_T^{\mathrm{cir}} \le \rho_2 h, \ \ \forall\, T \in \mathcal{T}_h, \ \ \forall\, 0 < h < 1, \tag{2.1}$$

where R_T^{inr} and R_T^{cir} are the inradius and the circumradius of T. Write

$$\xi_T = \frac{1}{\sqrt{|T|}} \int\limits_T 1 \, \mathrm{d}W(x)$$

for each triangle $T \in \mathcal{T}_h$, where $|T|$ denotes the area of T. It is well-known that $\{\xi_T\}_{T \in \mathcal{T}_h}$ is a family of independent and identically distributed normal random variables with mean 0 and variance 1 (see [22]). Then the piecewise constant approximation to $\dot{W}(x)$ is given by

$$\dot{W}_h(x) = \sum_{T \in \mathcal{T}_h} |T|^{-\frac{1}{2}} \xi_T \chi_T(x), \tag{2.2}$$

where χ_T is the characteristic function of T. It is apparent that $\dot{W}_h \in L^2(\Omega)$ almost surely. However, we have the following estimate which shows that $\|\dot{W}_h\|$ is unbounded as $h \to 0$.

Lemma 1 *There exist positive constants C_1 and C_2 independent of h such that*

$$C_1 h^{-2} \leq E\left(\|\dot{W}_h\|^2\right) \leq C_2 h^{-2}. \tag{2.3}$$

Proof It is easy to see that

$$E\left(\|\dot{W}_h\|^2\right) = \sum_{T\in\mathcal{T}_h} 1 = \sum_{T\in\mathcal{T}_h} |T|\frac{1}{|T|}.$$

By (2.1), $4\pi\rho_1^2 h^2 \leq |T| \leq 4\pi\rho_2^2 h^2$ for all $T \in \{\mathcal{T}_h\}$. Thus, we have

$$E\left(\|\dot{W}_h\|^2\right) \geq \frac{1}{4\pi\rho_2^2} h^{-2} \sum_{T\in\mathcal{T}_h} |T| = \frac{|\Omega|}{4\pi\rho_4^2} h^{-2},$$

$$E\left(\|\dot{W}_h\|^2\right) \leq \frac{1}{4\pi\rho_1^2} h^{-2} \sum_{T\in\mathcal{T}_h} |T| = \frac{|\Omega|}{4\pi\rho_1^2} h^{-2}.$$

Hence, (2.3) holds with $C_1 = \frac{|\Omega|}{4\pi\rho_2^2}$, $C_2 = \frac{|\Omega|}{4\pi\rho_1^2}$. □

Replacing $\dot{W}$ by $\dot{W}_h$ in (1.1), we have the following approximation problem:

$$\begin{cases} -\Delta u_h(x) + f(u_h(x)) = g(x) + \dot{W}_h(x), & x \in \Omega, \\ u_h = 0, & x \in \partial\Omega. \end{cases} \tag{2.4}$$

Its variational form is: find $u_h \in H_0^1(\Omega)$ such that

$$a(u_h, v) = (F_h, v), \quad \textit{for all } v \in H_0^1(\Omega), \tag{2.5}$$

where $F_h = g + \dot{W}_h$, and

$$a(\phi, \psi) = (\nabla\phi, \nabla\psi) + (f(\phi), \psi).$$

In the remaining of this section, we first show that (2.5) has a unique solution u_h in $H_0^1(\Omega) \cap H^2(\Omega)$ and then establish an estimate for the error $u - u_h$. To these ends, we shall assume that f satisfies the following conditions:

(A1) There is a constant $\alpha < \gamma$ such that

$$(f(s) - f(t))(s - t) \geq -\alpha|s - t|^2, \quad \forall\, s, t \in R.$$

(A2) There are positive constants β_1 and β_2 such that

$$|f(s) - f(t)| \leq \beta_1 + \beta_2|s - t|, \quad \forall\, s, t \in R.$$

Here γ is the positive constant in the Poincaré's inequality (see [1][10]):

$$\|\nabla v\|^2 \geq \gamma \|v\|^2, \quad \forall v \in H_0^1(\Omega). \tag{2.6}$$

These two conditions can be satisfied when f is a sum of a non-decreasing bounded function and a Lipschitz continuous function with the Lipschitz constant less than γ.

Theorem 1 *Under assumptions* ($A1$) *and* ($A2$), *the variational problem* (2.5) *has a unique solution in* $H_0^1(\Omega) \cap H^2(\Omega)$ *almost surely and*

$$E\left(\|u_h\|_2^2\right) \leq C_2 h^{-2}, \tag{2.7}$$

where $\|\cdot\|_2$ *denotes the norm of* $H^2(\Omega)$, C_2 *is a positive constant independent of* h.

Proof The existence of a unique solution $u_h \in H_0^1(\Omega)$ follows from Proposition 2.9 of [26]. By condition (A1) and the Poincaré's inequality (2.6), we obtain

$$a(\phi, \phi) \geq \|\nabla \phi\|^2 - \alpha\|\phi\|^2 \geq \frac{\gamma - \alpha}{1 + \gamma}\|\phi\|_1^2, \quad \forall \phi \in H_0^1(\Omega).$$

Thus, for $v = u_h$ in (2.5), we have

$$\|u_h\|_1^2 \leq \frac{1+\gamma}{\gamma - \alpha} a(u_h, u_h) = \frac{1+\gamma}{\gamma-\alpha}(F_h, u_h) \leq \frac{1+\gamma}{\gamma-\alpha}\|F_h\|\|u_h\|,$$

i.e.,

$$\|u_h\|_1 \leq \frac{1+\gamma}{\gamma-\alpha}\|F_h\|.$$

Let $R_h = -f(u_h) + F_h$. It follows from (A2) that

$$\|R_h\|^2 \leq 3\left(\beta_1^2|\Omega| + (\beta_2^2(1+\gamma)^2/(\gamma-\alpha)^2 + 1)\|F_h\|^2\right). \tag{2.8}$$

Notice that u_h is the unique weak solution of the boundary value problem

$$\begin{cases} -\Delta u_h = R_h, & \text{in } \Omega, \\ u_h = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

Therefore, by the results of the solution regularity of (2.9) (see [10]), we have that $u_h \in H^2(\Omega)$, and

$$\|u_h\|_2^2 \leq \rho_3 \|R_h\|^2 \leq 3\rho_3,$$

where ρ_3 is a positive constant only dependent on Ω. The estimate (2.7) follows from the above inequality, (2.3), and (2.8). □

Next we estimate the error between the weak solution u of (1.1) and its approximation u_h. Recall that u and u_h are the unique solutions of the following Hammerstein integral equations, respectively (see [6]):

$$u + Kf(u) = Kg + K\dot{W}, \tag{2.10}$$

$$u_h + Kf(u_h) = Kg + K\dot{W}_h, \tag{2.11}$$

where

$$K\phi(x) = \int_\Omega G(x,y)\phi(y)\mathrm{d}y$$

and $G(x,y)$ is the Green function of the Laplace equation with homogeneous Dirichlet boundary condition. It is well-known that

$$G(x,y) = -\frac{1}{2\pi}\log|x-y| + V(x,y) \tag{2.12}$$

where $V(x,y)$ is a Lipschitz continuous function of x and y (see Sect. 5.14 of [8]). We also have, by the Poincaré's inequality (2.6) (see Lemma 2.4 of [6]),

$$(K\phi, \phi) \geq \gamma \|K\phi\|^2, \quad \forall\, \phi \in L^2(\Omega). \tag{2.13}$$

The following lemma regarding the regularity of the Green function G defined in (2.12) will play an important role in our error estimates.

Lemma 2 *There exists a positive number ρ_4 independent of $\epsilon \in (0,1)$ such that*

$$\int_\Omega |G(x,y) - G(x,z)|^2 \mathrm{d}x \leq \rho_4 \epsilon^{-1} |y-z|^{2-\epsilon}, \quad \forall\, x, z \in \Omega. \tag{2.14}$$

Proof We only need to show that (2.14) holds for the singular part of G. For $0 < \epsilon < 1$, we have

$$
\begin{aligned}
&\int_\Omega (\log|x-y| - \log|x-z|)^2 \mathrm{d}x \\
&\quad = \int_\Omega (|x-y| - |x-z|)^{2-\epsilon} \big|\log|x-y| - \log|y-z|\big|^{\epsilon} \\
&\qquad \times \left(\int_0^1 \frac{\mathrm{d}\theta}{\theta|x-y| + (1-\theta)|x-z|} \right)^{2-\epsilon} \mathrm{d}x \\
&\quad \le |y-z|^{2-\epsilon} \int_\Omega \big|\log|x-y| - \log|y-z|\big|^{\epsilon} \\
&\qquad \times \left(\int_0^1 \frac{\mathrm{d}\theta}{\theta|x-y| + (1-\theta)|x-z|} \right)^{2-\epsilon} \mathrm{d}x \\
&\quad \le |y-z|^{2-\epsilon} \int_\Omega \big|\log|x-y| - \log|x-z|\big|^{\epsilon} \left(\frac{1}{|x-y|} + \frac{1}{|x-z|} \right)^{2-\epsilon} \mathrm{d}x.
\end{aligned}
$$

Using the Hölder inequality with $p = \frac{3}{\epsilon}$ and $q = \frac{3}{3-\epsilon}$ we have that

$$
\begin{aligned}
&\int_\Omega (\log|x-y| - \log|x-z|)^2 \mathrm{d}x \\
&\quad \le |y-z|^{2-\epsilon} \left(\int_\Omega \big|\log|x-y| - \log|x-z|\big|^3 \mathrm{d}x \right)^{\frac{\epsilon}{3}} \\
&\qquad \times \left(\int_\Omega \left(\frac{1}{|x-y|} + \frac{1}{|x-z|} \right)^{\frac{3(2-\epsilon)}{3-\epsilon}} \mathrm{d}x \right)^{\frac{3-\epsilon}{3}}.
\end{aligned}
$$

Let $H = \sup_{x,y\in\Omega} |x-y|$. Then we have

$$
\begin{aligned}
\int_\Omega \big|\log|x-y| - \log|x-z|\big|^3 &\le 2 \int_\Omega \left(\big|\log|x-y|\big|^3 + \big|\log|x-z|\big|^3 \right) \mathrm{d}x \\
&\le 4 \int_{|x-y|\le H} \big|\log|x-y|\big|^3 \mathrm{d}x \\
&= 8\pi \int_0^H r\big|\log(r)\big|^3 \mathrm{d}r
\end{aligned}
$$

and

$$\int_\Omega \left(\frac{1}{|x-y|} + \frac{1}{|x-z|} \right)^{\frac{3(2-\epsilon)}{3-\epsilon}} \mathrm{d}x \le 2^{\frac{3(2-\epsilon)}{3-\epsilon}} \int_{|x-y|\le H} \frac{1}{|x-y|^{\frac{3(2-\epsilon)}{3-\epsilon}}} \mathrm{d}x$$

$$\le 8\pi \int_0^H r^{-\frac{3-2\epsilon}{3-\epsilon}} \mathrm{d}r = 8\pi \frac{3-\epsilon}{\epsilon} H^{\frac{\epsilon}{3-\epsilon}} \le 24\pi\epsilon^{-1} H^{\frac{\epsilon}{3-\epsilon}}.$$

Combining the above inequalities, we obtain the desired estimate (2.14). □

Now we are in a position to establish an error estimate between u and u_h.

Theorem 2 *Let u and u_h be the solution of* (1.1) *and* (2.4)*, respectively. If f satisfies* ($A1$) *and* ($A2$)*, then there is a positive constant C_3 independent of u and h such that*

$$E\left(\|u-u_h\|^2\right) \le C_3\beta_1|\log(h)|^{\frac{1}{2}} h + C_4|\log(h)|h^2. \tag{2.15}$$

Proof Subtracting (2.11) from (2.10), we obtain

$$u(x) - u_h(x) + K(f(u) - f(u_h)) = E_h, \tag{2.16}$$

where $E_h = K\dot{W} - K\dot{W}_h$.

We first prove that there exists a positive constant C_5 independent of h such that

$$E\left(\|E_h\|^2\right) \le C_5|\log(h)|h^2. \tag{2.17}$$

Using the Ito isometry we have that

$$E\left(\|K\dot{W} - K\dot{W}_h\|^2\right)$$

$$= E\left(\int_\Omega \left[\int_\Omega G(x,y)\mathrm{d}W(y) - \int_\Omega G(x,y)\mathrm{d}W_h(y) \right]^2 \mathrm{d}x \right)$$

$$= E\left(\int_\Omega \left[\sum_{T\in\mathcal{T}_h} \int_T G(x,y)\mathrm{d}W(y) - |T|^{-1} \sum_{T\in\mathcal{T}_h} \int_T G(x,z)\mathrm{d}z \int_T 1\mathrm{d}W(y) \right]^2 \mathrm{d}x \right)$$

$$= E\left(\int_\Omega \left[\sum_{T\in\mathcal{T}_h} \int_T \int_T |T|^{-1}(G(x,y) - G(x,z))\mathrm{d}z\mathrm{d}W(y) \right]^2 \mathrm{d}x \right)$$

$$= \int_\Omega \left(\sum_{T\in\mathcal{T}_h} \int_T \left[|T|^{-1} \int_T (G(x,y) - G(x,z))\mathrm{d}z \right]^2 \mathrm{d}y \right) \mathrm{d}x.$$

From the Hölder inequality and (2.14) we obtain

$$
\begin{aligned}
E\left(\|K\dot{W}-K\dot{W}_h\|^2\right) &\le \int_\Omega \left(\sum_{T\in\mathcal{T}_h} |T|^{-1}\int_T\int_T (G(x,y)-G(x,z))^2 \mathrm{d}z\mathrm{d}y\right)\mathrm{d}x \\
&= \sum_{T\in\mathcal{T}_h} |T|^{-1}\int_T\int_T\int_\Omega (G(x,y)-G(x,z))^2 \mathrm{d}x\mathrm{d}z\mathrm{d}y \\
&\le \sum_{T\in\mathcal{T}_h} |T|^{-1}\int_T\int_T \rho_4\epsilon^{-1}|y-z|^{2-\epsilon}\mathrm{d}z\mathrm{d}y \\
&\le \rho_4|\Omega|\epsilon^{-1}h^{2-\epsilon}.
\end{aligned}
$$

Letting $\epsilon = 1/|\log(h)|$, we obtain (2.17) with $C_5 = e\rho_4|\Omega|$.

Multiplying (2.11) by $f(u)-f(u_h)$, we have

$$
(u-u_h, f(u)-f(u_h)) + (K(f(u)-f(u_h)), f(u)-f(u_h)) = (E_h, f(u)-f(u_h)).
$$

Then, by (2.13) and (A1), we obtain

$$
-\alpha\|u-u_h\|^2 + \gamma\|K(f(u)-f(u_h))\|^2 \le \|E_h\|\|f(u)-f(u_h)\|. \tag{2.18}
$$

Then, from (2.16), we have

$$
\|K(f(u)-f(u_h))\|^2 = \|u-u_h-E_h\|^2 \ge \frac{\alpha+\gamma}{2\gamma}\|u-u_h\|^2 - \frac{3\gamma-\alpha}{\gamma-\alpha}\|E_h\|^2, \tag{2.19}
$$

where we have used the following inequality with $\epsilon = (\alpha+\gamma)/(2\gamma)$:

$$
\|\phi+\psi\|^2 \ge \epsilon\|\phi\|^2 - \frac{2-\epsilon}{1-\epsilon}\|\psi\|^2, \quad \forall\, 0<\epsilon<1,\ \phi,\psi\in L^2(\Omega).
$$

It follows from (A2) that

$$
\|f(u)-f(u_h)\| \le \beta_1|\Omega|^{\frac{1}{2}} + \beta_2\|u-u_h\|.
$$

Thus,

$$
\begin{aligned}
\|E_h\|\|f(u)-f(u_h)\| &\le \beta_1|\Omega|^{\frac{1}{2}}\|E_h\| + \beta_2\|u-u_h\|\|E_h\| \\
&\le \beta_1|\Omega|^{\frac{1}{2}}\|E_h\| + \frac{\alpha-\gamma}{4}\|u-u_h\|^2 + \frac{\beta_2^2}{\alpha-\gamma}\|E_h\|^2.
\end{aligned} \tag{2.20}
$$

Combining (2.18)–(2.20), we obtain

$$
\|u-u_h\|^2 \le C_6\beta_1\|E_h\| + C_7\|E_h\|^2,
$$

where C_6 and C_7 are positive constants independent of β_1, u, u_h, and h. Then (2.15) follows from the above inequality and (2.17) □

3 Finite element methods

In this section, we consider the finite element approxiamtions of variational problem (2.5) and establish their error estimates.

Let V_h be a linear finite element subspace of $H_0^1(\Omega)$ with respect to the triangulation $\mathcal{T}_h$ specified in Sect. 2. Then the finite element approximation to (2.4) is: Find $U_h \in V_h$ such that

$$(\nabla U_h, \nabla v) + (f(U_h), v) = (g + \dot{W}_h, v), \quad \forall\, v \in V_h. \tag{3.1}$$

We have the following theorem about the existence of a unique solution U_h of (3.1).

Theorem 3 *If f satisfies $(A1)$ and $(A2)$, the approximate variational problem (3.1) has a unique solution U_h. In addition, there is a positive constant C_8 such that*

$$E(\|U_h\|_1^2) \le C_8 h^{-2}. \tag{3.2}$$

Proof The proof is the same as the first part of the proof of Theorem 1. □

In order to estimate the error $u_h - U_h$, we need the Galerkin projection operator $P_h :\ H_0^1(\Omega) \longrightarrow V_h$ defined by

$$(\nabla P_h w, \nabla v) = (\nabla w, \nabla v), \quad \forall\, v \in V_h,\ \ w \in H_0^1(\Omega).$$

It is well-known that (see [5])

$$\|w - P_h w\| + h\|\nabla(w - P_h w)\| \le \rho_5 h^2 \|w\|_2, \quad \forall\, w \in H^2(\Omega) \cap H_0^1(\Omega), \tag{3.3}$$

where ρ_5 is a positive constant independent of h.

Theorem 4 *If f satisfies $(A1)$ and $(A2)$, then there is a positive constant C_6 such that*

$$E(\|u - U_h\|^2) \le C_9 |\log(h)|^{\frac{1}{2}} h. \tag{3.4}$$

Proof It is easy to see that

$$(\nabla(P_h u_h - U_h), \nabla(P_h u_h - U_h)) + (f(u_h) - f(U_h), P_h u_h - U_h) = 0.$$

Thus, we have by (A1) and (A2)

$$
\begin{aligned}
&\|\nabla(P_h u_h - U_h)\|^2 \\
&\quad = -(f(u_h) - f(U_h), u_h - U_h) + (f(u_h) - f(U_h), u_h - P_h u_h) \\
&\quad \le \alpha\|u_h - U_h\|^2 + \|\beta_1 + \beta_2|u_h - U_h|\|\|u_h - P_h u_h\| \\
&\quad \le \alpha\|u_h - U_h\|^2 + \beta_1|\Omega|^{\frac{1}{2}}\|u_h - P_h u_h\| + \beta_2\|u_h - U_h\|\|u_h - P_h u_h\| \\
&\quad \le \frac{\gamma+\alpha}{2}\|u_h - U_h\|^2 + \beta_1|\Omega|^{\frac{1}{2}}\|u_h - P_h u_h\| + \frac{\beta_2^2}{2(\gamma-\alpha)}\|u_h - P_h u_h\|^2
\end{aligned}
$$

By Poicaré inequality (2.6), we have

$$
\begin{aligned}
\gamma\|u_h - U_h\|^2 &\le \gamma\|u_h - P_h u_h\|^2 + \gamma\|P_h u_h - U_h\|^2 \\
&\le \gamma\|u_h - P_h u_h\|^2 + \|\nabla(P_h u_h - U_h)\|^2 \\
&\le \frac{\gamma+\alpha}{2}\|u_h - U_h\|^2 + \beta_1|\Omega|^{\frac{1}{2}}\|u_h - P_h u_h\| \\
&\quad + \left(\gamma + \frac{\beta_2^2}{2(\gamma-\alpha)}\right)\|u_h - P_h u_h\|^2
\end{aligned}
$$

Hence, we obtain by (3.3)

$$
\|u_h - U_h\|^2 \le C_{10}(\beta_1 h^2\|u_h\|_2 + h^4\|u_h\|_2^2),
$$

where C_{10} is a positive constant independent on h and β_1. By Theorem 1 we have

$$
\begin{aligned}
E\left(\|u_h - U_h\|^2\right) &\le C_{10}\left(\beta_1 h^2 E\left(\|u_h\|_2\right) + h^4 E\left(\|u_h\|_2^2\right)\right) \\
&\le C_{10}\left(\beta_1 h^2 E\left(\|u_h\|_2^2\right)^{\frac{1}{2}} + h^4 E\left(\|u_h\|_2^2\right)\right) \\
&\le C_{10}\left(\beta_1 C_1^{\frac{1}{2}} h + C_1^2 h^2\right). \qquad (3.5)
\end{aligned}
$$

Then by Theorem 2, we obtain

$$
E\left(\|u - U_h\|^2\right) \le C_5\beta_1|\log(h)|^{\frac{1}{2}} h + C_6|\log(h)|h^2 + C_{10}\left(\beta_1 C_1^{\frac{1}{2}} h + C_1^2 h^2\right), \qquad (3.6)
$$

which leads to the error estimate (3.4). □

Remark 1 In [12], Gyongy and Martinez studied the finite difference approximation of (1.1). They proved that

$$
E(\|u - U_h^d\|^2) \le Ch^\gamma \qquad (3.7)
$$

where U_h^d is the finite difference solution for (1.1) and $\gamma < 1$. Clearly, our error estimate is comparable to theirs. In fact, using the technique in this paper, the error estimate (3.7) can be expressed exactly the same as given in (3.4).

It should be pointed that we also have

$$E(\|u_h\|_1^2) \leq C_8 h^{-2}.$$

So u_h and its finite element approximation U_h have the same bound of order h^{-2}. Although u_h and U_h are unbounded in $H_0^1(\Omega)$, we still have a positive order estimate of the expectation of the error $u_h - U_h$ in $L^2(\Omega)$ as shown in (3.5). When the nonlinear function f in (1.1) is Lipschitz continuous, we have a much strong error estimate.

Theorem 5 *If f is Lipschitz continuous with the Lipschitz constant L less than γ, then there is a positive constant C_{11} independent on h such that*

$$E(\|u - U_h\|^2) \leq C_{11} |\log(h)| h^2. \tag{3.8}$$

Proof In this case, Assumption (A1) and Assumption (A2) hold for $\beta_1 = 0$, $\beta_2 = \alpha = L$. Then (3.8) follows from (3.6). □

Remark 2 The estimate (3.8) is optimal with respect to the regularity estimate of $E(\|u_h\|_2^2)$. A direct consequence of Theorem 5 is the error estimate when (1.1) is a linear SPDE, i.e., $f(u) = c(x)u$, where $c(x) \in L^\infty(\Omega)$ has $-\alpha$ as its lower bound.

Remark 3 The above methodology can also be applied to the one dimensional case ($\Omega = (a, b)$) to generalize the results of [2] and [9] to nonlinear problems.

Remark 4 We should not expect any estimate of $E\left(\|\nabla(u_h - U_h)\|^2\right)$ with a positive order since $E(\|u_h\|_2^2)$ is of order -2. However, by the proof of Theorem 4, there is a positive constant C_{12} independent on h and β_1 such that

$$E\left(\|\nabla(P_h u_h - U_h)\|^2\right) \leq C_{12}(\beta_1 h + h^2), \tag{3.9}$$

which agrees with the property of superconvergence of finite element methods.

4 Numerical experiments

In this section, we present numerical examples to demonstrate our theoretical results in the previous section. We will consider both linear and nonlinear problems, first on the unit square and then on the unit disc.

The normal random variables for $\dot{W}_h$ shall be simulated by using the random number generator *gsl_ran_gaussian* of the GNU Scientific Library (GSL). Theoretically, the number of samples M should be chosen so that the error generated by the Monte Carlo method is in the same magnitude of the errors generated

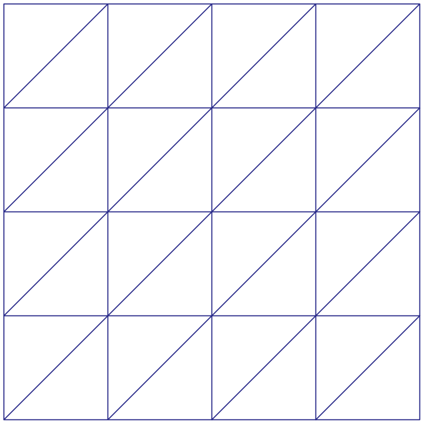

Fig. 1 The partition of the unit square for $N = 4$

by the finite element approximation. Although for linear problem, $E(U_h)$ is the finite element approximation of the deterministic solution, we shall evaluate $E(U_h)$ by using the Monte Carlo method to examine

$$e_1(h) = \|E(u) - E(U_h)\|$$

to see if we have used enough samples. We also employ the following two types of errors

$$e_2(h) = |E(\|u\|^2) - E(\|U_h\|^2)|,$$
$$e_3(h) = |E(\|U_h\|^2) - E(\|U_{\frac{h}{2}}\|^2)|$$

to check our theoretical error estimates for linear and nonlinear problems, respectively. Obviously these two errors together with e_1 can be controlled by the error $(E(\|u - U_h\|^2))^{\frac{1}{2}}$, but not equivalent to it. Nevertheless we believe that they provide good indications about how the error $(E\|u - U_h\|^2)^{\frac{1}{2}}$ itself behaves.

Example 1 In this example, we take Ω to be the unit square, i.e., $\Omega = (0, 1) \times (0, 1)$. Let the exact solution be $\overline{u}(x, y) = \sin(\pi x)\sin(\pi y)$ in the absence of the white noise. The unit square will be triangulated as shown in Fig. 1. We have that $h = \sqrt{2}/N = 0.3535534, 0.176777, 0.088388, 0.0441942, 0.0220971$ for $N = 4, 8, 16, 32, 64$.

First, consider the linear problem, i.e., $f(u) = 0$. In this case, we have that $E(u) = \overline{u}$ and $\|E(u)\|^2 = 0.25$. Recall that (see [8,11])

$$G(x, y) = \frac{4}{\pi^2} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{1}{(p + q)^2} \sin(p\pi x)\sin(q\pi y).$$

Table 1 Linear problem on the unit square: Test 1

M	N	h	e_1	Rate	$E(\|U_h\|^2)$	e_2	Rate
2000	4	0.353553	7.95e–2		0.1867162	6.82e–2	
8000	8	0.176776	2.10e–2	1.92	0.2351932	1.97e–2	1.79
32000	16	0.088388	5.41e–3	1.96	0.2492230	5.69e–3	1.79
128000	32	0.044194	1.43e–3	1.91	0.2529695	1.95e–3	1.55

Table 2 Linear problem on the unit square: Test 2

M	N	h	e_1	Rate	$E(\|U_h\|^2)$	e_2	Rate
2000	4	0.353553	7.88e–2		0.1875694	6.73e–2	
8000	8	0.176776	2.14e–2	1.88	0.2347563	2.02e–2	1.74
32000	16	0.088388	5.74e–3	1.90	0.2488673	6.05e–3	1.74
128000	32	0.044194	1.50e–3	1.93	0.2529045	2.01e–3	1.59

Table 3 Linear problem on the unit square: Test 3

M	N	h	e_1	Rate	$E(\|U_h\|^2)$	e_2	Rate
2000	4	0.353553	7.87e–2		0.1876089	6.73e–2	
8000	8	0.176776	2.15e–2	1.87	0.2346670	2.02e–2	1.73
32000	16	0.088388	5.49e–3	1.97	0.2491416	5.78e–3	1.81
128000	32	0.044194	1.41e–3	1.96	0.2530078	1.91e–3	1.60

It is easy to see from the Ito's isometry that

$$E(\|u\|^2) = \|E(u)\|^2 + \int_\Omega \int_\Omega G(x,y)^2 \mathrm{d}x\mathrm{d}y.$$

By simple calculation, we obtain

$$E(\|u\|^2) = 0.25 + \frac{4}{\pi^4} \sum_{n=2}^{\infty} \frac{n-1}{n^4} = 0.25491673490338.$$

The computational results are displayed in Tables 1, 2, and 3. The numbers of samples, M, are displayed in the first columns of the tables. The third and forth columns of the tables show that the rate of convergence for $E(U_h)$ is of order 2 as expected, which implies that our sample sizes are good enough to ensure the accuracy of the Monte Carlo method. We observe from the last three columns that our finite element method converges faster than the theoretical rate $O(h \log(h))$. Notice that the rates on the last rows for e_2 are significantly smaller than others. We believe that this is due to the sample error rather than the finite element error.

Table 4 Nonlinear problem on the unit square: Test 1

M	N	h	$E(\|U_h\|^2)$	e_3	Rate
1000	4	0.353553	0.18951426		
4000	8	0.176776	0.23499990	4.5486e–2	
16000	16	0.088388	0.24867523	1.3675e–2	1.73
64000	32	0.044194	0.25243723	3.7620e–3	1.86

Table 5 Nonlinear problem on the unit square: Test 2

M	N	h	$E(\|U_h\|^2)$	e_3	Rate
1000	4	0.353553	0.19040172		
4000	8	0.176776	0.23525031	4.4849e–02	
16000	16	0.088388	0.24881669	1.3566e–02	1.73
64000	32	0.044194	0.25251834	3.7016e–03	1.87

Table 6 Nonlinear problem on the unit square: Test 3

M	N	h	$E(\|U_h\|^2)$	e_3	Rate
1000	4	0.353553	0.18892402		
4000	8	0.176776	0.23617167	4.7248e–2	
16000	16	0.088388	0.24945892	1.3287e–2	1.83
64000	32	0.044194	0.25248505	3.0261e–3	2.13

Next, let us consider the nonlinear problem with $f(u) = \sin(u)$. The results are displayed for three tests in Tables 4, 5, and 6. Again, we observe that our finite element method converges faster than the theoretical rate $O(h\log(h))$.

Example 2 In this example, let Ω be the unit disc and the exact solution is $\overline{u}(x) = \sin(\pi|x|^2)$ in the absence of white noise. The unit disc is initially triangulated by "Triangle" (by Jonathan Richard Shewchuk at http://www.cs.cmu.edu/ quake/ triangle.html) and then the mesh is optimized by "Meshgen" (by Lili Ju at http://www.math.sc.edu/~ju/). The triangulation with $h = 0.312869$ is depicted in Fig. 2.

For the linear problem ($f(u) = 0$), we have that $E(u) = \overline{u}$ and $\|E(u)\|^2 = \frac{\pi}{2}$. It is well-known that

$$G(x,y) = \frac{1}{4\pi}\log\frac{1+|x|^2|y|^2-2x\cdot y}{|x-y|^2}.$$

Then we obtain numerically

$$E(\|u\|^2) = 1.62016354682162.$$

From Tables 7, 8, 9, 10, 11, and 12, we have the same observations as in Example 1 for the linear problem ($f(u) = 0$) and the nonlinear problem ($f(u) = \sin(u)$).

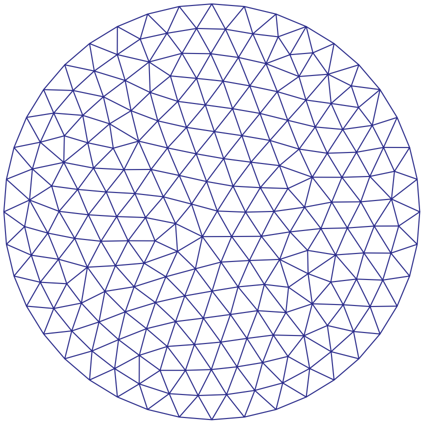

Fig. 2 The partition of the unit disc

Table 7 Linear problem on the unit disc: Test 1

M	h	e_1	Rate	$E(\|U_h\|^2)$	e_2	Rate
1000	0.312869	1.43e–1		1.3450300	2.75e–1	
4000	0.180526	4.66e–2	2.04	1.5314252	8.87e–2	2.06
16000	0.091617	1.17e–2	2.04	1.5991079	2.11e–2	2.12
64000	0.046305	3.09e–3	1.95	1.6158970	4.27e–3	2.34

Table 8 Linear problem on the unit disc: Test 2

M	h	e_1	Rate	$E(\|U_h\|^2)$	e_2	Rate
1000	0.312869	1.46e–1		1.3327223	2.87e–1	
4000	0.180526	4.57e–2	2.12	1.5377359	8.24e–2	2.27
16000	0.091617	1.20e–2	1.97	1.5976269	2.25e–2	1.91
64000	0.046305	3.19e–3	1.94	1.6138093	6.35e–3	1.86

Table 9 Linear problem on the unit disc: Test 3

M	h	e_1	Rate	$E(\|U_h\|^2)$	e_2	Rate
1000	0.312869	1.45e–1		1.3371378	2.83e–1	
4000	0.180526	4.61e–2	2.08	1.5391690	8.10e–2	2.28
16000	0.091617	1.17e–2	2.02	1.5991064	2.11e–2	1.99
64000	0.046305	2.86e–3	2.07	1.6153823	4.78e–3	2.17

5 Conclusions

Our aim in this work is to develop the finite element method for a class of semilinar elliptic stochastic differential equations driven by additive white noises. The previously published works in this area that we are aware of are [2,9], in which a one dimensional linear problem was studied. In this paper, we substantially

Table 10 Nonlinear problem on the unit disc: Test 1

M	h	$E(\|U_h\|^2)$	e_3	Rate
1000	0.312869	1.33172989		
4000	0.180526	1.53021581	1.9849e–1	
16000	0.091617	1.59739334	6.7178e–2	1.60
64000	0.046305	1.60845636	1.1063e–2	2.64

Table 11 Nonlinear problem on the unit disc: Test 2

M	h	$E(\|U_h\|^2)$	e_3	Rate
1000	0.312869	1.34696047		
4000	0.180526	1.53195882	1.8500e–1	
16000	0.091617	1.59393834	6.1980e–2	1.61
64000	0.046305	1.60783819	1.3900e–2	2.19

Table 12 Nonlinear problem on the unit disc: Test 3

M	h	$E(\|U_h\|^2)$	e_3	Rate
1000	0.312869	1.35172754		
4000	0.180526	1.52685297	1.7513e–1	
16000	0.091617	1.59138836	6.4535e–2	1.47
64000	0.046305	1.60794464	1.6556e–2	1.99

extend their work from one dimension to two dimension and from linear problems to nonlinear problems. More importantly, we allow the domain to be any convex set with regular boundary, not just a rectangle, which is the main advantage of the finite element method over other methods such as finite difference methods and spectral finite element methods. Both our theoretical analysis and numerical experiments establish the rates of convergence of the finite element approximate solutions. These rates provide a theoretical basis in determining the numbers of samples in Monte Carlo simulation for the discretized problems. Some of the interesting extensions of the current work include more efficient numerical simulations for white noise and numerical approximations for SPDEs with general nonlinear terms and with general random forcing terms.

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