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Error analysis of finite element approximations of the stochastic Stokes equations

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Abstract Numerical solutions of the stochastic Stokes equations driven by white noise perturbed forcing terms using finite element methods are considered. The discretization of the white noise and finite element approximation algorithms are studied. The rate of convergence of the finite element approximations is proved to be almost first order ($h|\ln h|$) in two dimensions and one half order ($h^{\frac{1}{2}}$) in three dimensions. Numerical results using the algorithms developed are also presented.

Keywords Stokes equations · White noise · Finite element methods

Mathematics Subject Classifications (2000) 60H15 · 65M60 · 35Q30

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1 Introduction

We consider the steady-state Stokes equation with the forcing term perturbed by white noise:

$$\begin{cases} -\nu\Delta u + \nabla p = f + \sigma \dot{W}, & \text{in } \Omega, \\ \operatorname{div} u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a subset of R^d for $d = 2, 3$, $u : \Omega \to R^d$ is the velocity of the fluid flow, p is the pressure, σ is a positive continuous function in Ω, $f \in L^2(\Omega)$ and $\dot{W} = (\dot{W}^1, \ldots, \dot{W}^d)$ is the white noise such that

$$E(\dot{W}^j(x)\dot{W}^j(x')) = \delta(x - x'), \quad x,\ x' \in \Omega, \quad j = 1, \ldots, d,$$

where δ denotes the usual Dirac delta function and E the expectation. We assume that Ω is a convex polygon in R^2 or a convex polyhedron in R^3. We also assume that $\dot{W}^i$ and $\dot{W}^j$, $i \neq j$, $i, j = 1, \ldots d$, are independent. Equation (1) is the linearization of the Navier-Stokes equation which describes the motion of an incompressible viscous fluid. The noise term occurs, for example, when there are uncertainties, such as the unknown temperature of the fluid, in the model. Success in solving (1) may help us understand and quantify uncertainties in processes described by (1).

Stochastic Navier-Stokes equations have recently attracted attentions from mathematical communities in both theoretical analysis and numerical simulation aspects [3, 4, 6, 8, 11]. In [11], the polynomial chaos method is used to study the numerical simulation of stochastic Navier-Stokes equation with viscosity as a random field. Polynomial chaos expansion is also used in [6] to study numerical solutions of stochastic Navier Stokes equations with white noise forcing terms.

The purpose of this paper is to conduct error analysis on finite element approximations for the solution (u, p) of (1). Following the general framework of [2], we first discretize the white noise using a tensor product between Gaussian random variables and characteristic functions of the triangular elements in the triangulations of Ω. Then we apply the standard finite element method to the stochastic Stokes equations with the white noise $\dot{W}$ in (1) replaced by the discretized white noise. Our error estimates show that the finite element approximation error for the velocity u is almost first order ($h|\log h|$) for $d = 2$ and one half ($h^{\frac{1}{2}}$) for $d = 3$. Here we emphasize that because of the lack of regularity, the convergence rate of the finite element solution for the stochastic Stokes equation is one order lower than that in the deterministic case for $d = 2$ and one and half order lower for $d = 3$.

The paper is organized as follows. In Section 2, we define the weak solution of (1) using Green's functions and study the approximation errors for the stochastic Stokes equations with discretized white noise forcing terms. In Section 3, we construct finite element approximations for the stochastic Stokes equations with discretized white noise forcing terms and carry out the error analysis. Finally, in Section 4, we present numerical simulation results using the algorithm constructed in Section 3.

2 Discretization of the white noise and error estimates

2.1 Weak solutions and properties of the Green's functions

In this subsection we use the Green's functions of the Stokes equation to define the weak solution of (1). First we introduce some notations which will be used throughout the rest of the paper. For $v = (v_1, \cdots, v_n) \in R^n$, denote $|v|^2 = \sqrt{v_1^2 + \cdots v_n^2}$. Also we use $\| \cdot \|$ to denote the norm in $L^2(\Omega)$ and $\| \cdot \|_k$ to denote the usual norm in Sobolev spaces $H^k(\Omega)$ for $k \in R$.

The Green's functions for the Stokes equation are given by $G = (G_{ij})_{d\times d}$ and $H = (H_j)_{1\times d}$ for u and p, respectively (see, e.g., [7]), where

$$G_{ij}(x, y) = -\frac{1}{4\pi}\nu\delta_{ij}\ln\frac{1}{|x-y|} + \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^2} + g_{ij}^1(x, y),\ i, j = 1, 2, \quad (2)$$

$$H_j(x, y) = \frac{\partial}{\partial x_j}D_j(x, y),\quad D_j(x, y) = -\frac{1}{2\pi}\ln\frac{1}{|x-y|} + h_j^1(x, y),\ j = 1, 2 \quad (3)$$

for $d = 2$ and

$$G_{ij}(x, y) = -\frac{1}{8\pi\nu}\delta_{ij}\frac{1}{|x-y|} + \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^3} + g_{ij}^2(x, y),\ i, j = 1, 2, 3, \quad (4)$$

$$H_j(x, y) = \frac{\partial}{\partial x_j}D_j(x, y),\quad D_j(x, y) = \frac{1}{4\pi}\frac{1}{|x-y|} + h_j^2(x, y),\ j = 1, 2, 3 \quad (5)$$

for $d = 3$. Here g_{ij}^k and h_j^k are piecewise differentiable functions on $\bar{\Omega}$. To simplify the presentation we assume that $\sigma \equiv 1$. We define the weak solution of (1) as follows.

$$u(x) = \int_\Omega G(x, y)f(y)dy + \int_\Omega G(x, y)dW(y) \quad (6)$$

and

$$p(x) = \int_\Omega H(x, y)f(y)dy + \operatorname{div}\int_\Omega D(x, y)dW(y) \quad (7)$$

where the stochastic integral is defined in Ito's sense. From Ito's isometry, it is easy to see that $u \in L^2(\Omega)$ almost surely. We will treat p as a function in $H^{-1}(\Omega)$. In fact it can be easily show that

$$E(\|p\|_{-1}^2) \le C\|D\|^2.$$

To study the approximation of (1) with discretized white noises, we need the following properties concerning the regularity of the Green's functions.

Lemma 1 *There exists a constant C such that if $|y - z|$ is sufficiently small, then*

$$\int_\Omega |G(x, y) - G(x, z)|^2 dx \le C|\ln(|y - z|)||y - z|^2, \tag{8}$$

$$\int_\Omega |D(x, y) - D(x, z)|^2 dx \le C|\ln(|y - z|)||y - z|^2 \tag{9}$$

for $d = 2$ and

$$\int_\Omega |G(x, y) - G(x, z)|^2 dx \le C|y - z|, \tag{10}$$

$$\int_\Omega |D(x, y) - D(x, z)|^2 dx \le C|y - z| \tag{11}$$

for $d = 3$.

Proof We only prove (8) and (10). The proofs of (9) and (11) are similar. It is proved in [2] that

$$\int_\Omega (\ln|x - y| - \ln|x - z|)^2 dx \le C|\ln(|y - z|)||y - z|^2. \tag{12}$$

Thus to prove (8), it suffices to prove that

$$\int_\Omega \left| \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^2} - \frac{(x_i - z_i)(x_j - z_j)}{|x - z|^2} \right|^2 dx \le C|\ln(|y - z|)||y - z|^2 \tag{13}$$

for $i, j = 1, 2$. We only prove (13) for $i = 1$ and $j = 2$. The proofs for the other cases are similar. Let $S(v) = \{x;\ |x - v| \le |y - z|\}$ and $\Omega_1 = \Omega \setminus (S(y) \cup S(z))$. Since the integrand in (13) is a bounded function, it suffices to prove (13) with Ω replaced by Ω_1. Simple calculations give

$$\begin{aligned}
&\int_{\Omega_1} \left(\frac{(x_1 - y_1)(x_2 - y_2)}{|x - y|^2} - \frac{(x_1 - z_1)(x_2 - z_2)}{|x - z|^2} \right)^2 dx \\
&= \int_{\Omega_1} \left(\frac{(x_1 - y_1)(x_2 - y_2)|x - z|^2 - (x_1 - z_1)(x_2 - z_2)|x - y|^2}{|x - y|^2|x - z|^2} \right)^2 dx \\
&\le 2\int_{\Omega_1} \left(\frac{|x - z|^2((x_1 - y_1)(x_2 - y_2) - (x_1 - z_1)(x_2 - z_2))}{|x - z|^2|x - y|^2} \right)^2 dx \\
&\quad + 2\int_{\Omega_1} \left(\frac{(|x - z|^2 - |x - y|^2)(x_1 - z_1)(x_2 - z_2)}{|x - z|^2|x - y|^2} \right)^2 dx =: I_1 + I_2.
\end{aligned}$$

For I_1 we have that

$$
\begin{aligned}
I_1 &= 2\int_{\Omega_1}\left(\frac{(x_1-y_1)(x_2-y_2)-(x_1-z_1)(x_2-z_2)}{|x-y|^2}\right)^2 dx\\
&= 2\int_{\Omega_1}\left(\frac{(x_2-z_2)(y_1-z_1)+(x_1-y_1)(y_2-z_2)}{|x-y|^2}\right)^2 dx\\
&\le 2|y-z|^2\int_{\Omega_1}\left(\frac{|x-z|^2+|x-y|^2}{|x-y|^4}\right) dx\\
&\le 2|y-z|^2\int_{\Omega_1}\left(\frac{2|y-z|^2}{|x-y|^4}+\frac{3}{|x-y|^2}\right) dx\\
&= 4|y-z|^4\int_{\Omega_1}\frac{1}{|x-y|^4}dx+6|y-z|^2\int_{\Omega_1}\frac{1}{|x-y|^2}dx.
\end{aligned}
$$

Since Ω is bounded, there exists a constant R such that $|x|\le R$ for $x\in\Omega$. Thus

$$
\begin{aligned}
I_1 &\le 8\pi|y-z|^4\int_{|y-z|}^{2R}\frac{1}{r^3}dr+12\pi|y-z|^2\int_{|y-z|}^{2R}\frac{1}{r}dr\\
&\le C(|y-z|^2+|y-z|^2|\ln(|y-z|)|).
\end{aligned}
$$

Using a similar argument, we can also derive the following estimate.

$$
I_2\le C(|y-z|^2+|y-z|^2|\ln(|y-z|)|).
$$

This proves (13), thus (8). Next we prove (10). To this end it suffices to prove that

$$
\int_{\Omega}\left|\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^3}-\frac{(x_i-z_i)(x_j-z_j)}{|x-z|^3}\right|^2 dx\le C|y-z| \tag{14}
$$

for $i,j=1,2,3$ and

$$
\int_{\Omega}\left|\frac{1}{|x-y|}-\frac{1}{|x-z|}\right|^2 dx\le C|y-z|. \tag{15}
$$

We first prove (14). We only consider the case when $i=1$ and $j=2$ since the proofs for the other cases are similar. Let $S(v)=\{x;\ |x-v|\le|y-z|\}$ and $\Omega_1=\Omega\setminus(S(y)\cup S(z))$. We estimate the integral on the left hand side of (14) on $(S(y)\cup S(z))$ and Ω_1 separately. Note that

$$
\begin{aligned}
\int_{S(y)\cup S(z)}\left[\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^3}\right]^2 dx &\le \int_{S(y)\cup S(z)}\left[\frac{(x_1-y_1)^2+(x_2-y_2)^2}{|x-y|^3}\right]^2 dx\\
&\le \int_{S(y)\cup S(z)}\frac{1}{|x-y|^2}dx.
\end{aligned}
$$

Using the spherical coordinates, we have that

$$\int_{S(y)} \frac{1}{|x-y|^2} dx = \int_0^{|y-z|} \int_0^{2\pi} \int_0^{\pi} \frac{r^2 \sin\phi}{r^2} d\phi d\theta dr = 4\pi|y-z|.$$

On the other hand,

$$\begin{aligned}\int_{(S(y)\cup S(z))\setminus S(y)} \frac{1}{|x-y|^2} dx &\leq \int_{(S(y)\cup S(z))\setminus S(y)} \frac{1}{|x-z|^2} dx \\ &\leq \int_{S(z)} \frac{1}{|x-z|^2} dx = 4\pi|y-z|.\end{aligned}$$

This proves that

$$\int_{S(y)\cup S(z)} \left[\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^3}\right]^2 dx \leq 8\pi|y-z|.$$

Similarly we have that

$$\int_{S(y)\cup S(z)} \left[\frac{(x_1-z_1)(x_2-z_2)}{|x-z|^3}\right]^2 dx \leq 8\pi|y-z|.$$

Thus

$$\begin{aligned}&\int_{S(y)\cup S(z)} \left[\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^3} - \frac{(x_1-z_1)(x_2-z_2)}{|x-z|^3}\right]^2 dx \\ &\leq 2\int_{S(y)\cup S(z)} \left(\left[\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^3}\right]^2 + \left[\frac{(x_1-z_1)(x_2-z_2)}{|x-z|^3}\right]^2\right) dx \leq 16\pi|y-z|.\end{aligned}$$

Now we turn to the estimate of the integral on Ω_1. Simple calculations give

$$\begin{aligned}&\int_{\Omega_1} \left(\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^3} - \frac{(x_1-z_1)(x_2-z_2)}{|x-z|^3}\right)^2 dx \\ &= \int_{\Omega_1} \left(\frac{(x_1-y_1)(x_2-y_2)|x-z|^3 - (x_1-z_1)(x_2-z_2)|x-y|^3}{|x-y|^3|x-z|^3}\right)^2 dx \\ &\leq 2\int_{\Omega_1} \left(\frac{|x-z|^3((x_1-y_1)(x_2-y_2) - (x_1-z_1)(x_2-z_2))}{|x-z|^3|x-y|^3}\right)^2 dx \\ &\quad + 2\int_{\Omega_1} \left(\frac{(|x-z|^3 - |x-y|^3)(x_1-z_1)(x_2-z_2)}{|x-z|^3|x-y|^3}\right)^2 dx =: I_1 + I_2.\end{aligned}$$

For I_1 we have that

$$\begin{aligned}
I_1 &= 2\int_{\Omega_1}\left(\frac{(x_1-y_1)(x_2-y_2)-(x_1-z_1)(x_2-z_2)}{|x-y|^3}\right)^2 dx\\
&= 2\int_{\Omega_1}\left(\frac{(x_2-z_2)(y_1-z_1)-(x_1-y_1)(y_2-z_2)}{|x-y|^3}\right)^2 dx\\
&\le 2|y-z|^2\int_{\Omega_1}\left(\frac{|x-z|^2+|x-y|^2}{|x-y|^6}\right)dx\\
&\le 2|y-z|^2\int_{\Omega_1}\left(\frac{2|y-z|^2}{|x-y|^6}+\frac{3}{|x-y|^4}\right)dx\\
&= 4|y-z|^4\int_{\Omega_1}\frac{1}{|x-y|^6}dx+6|y-z|^2\int_{\Omega_1}\frac{1}{|x-y|^4}dx.
\end{aligned}$$

Since Ω is bounded, there exists a constant R such that $|x|\le R$ for $x\in\Omega$. Thus

$$\begin{aligned}
I_1 &\le 16\pi|y-z|^4\int_{|y-z|}^{2R}\frac{1}{r^4}dr+24\pi|y-z|^2\int_{|y-z|}^{2R}\frac{1}{r^2}dr\\
&\le C|y-z|.
\end{aligned}$$

Using a similar argument, we can also derive the following estimate.

$$I_2\le C|y-z|$$

which, together with the estimate for I_1, proves (14). Next we prove (15). Using the Hölder inequality we have that

$$\begin{aligned}
&\int_{\Omega_1}\left(\frac{1}{|x-y|}-\frac{1}{|x-z|}\right)^2 dx\\
&\quad\le\int_{\Omega_1}\frac{|y-z|^2}{|x-y|^2|x-z|^2}dx\\
&\quad\le\frac{|y-z|^2}{2}\int_{\Omega_1}\left(\frac{1}{|x-y|^4}+\frac{1}{|x-z|^4}\right)dx\\
&\quad\le|y-z|^2\int_{|y-z|}^{R}\frac{1}{r^2}dr\\
&\quad\le C|y-z|.
\end{aligned}$$

For the integral in $S(y)\cup S(z)$, we follow the first part of the proof of (14) to obtain

$$\int_{S(y)\cup S(z)}\left(\frac{1}{|x-y|}-\frac{1}{|x-z|}\right)^2 dx\le\int_{S(y)\cup S(z)}\left(\frac{1}{|x-y|^2}+\frac{1}{|x-z|^2}\right)dx\le 16\pi|y-z|.$$

This proves (15) and thus (10). The proof is complete. □

2.2 Approximation with discretized white noise

In this subsection we define an approximate solution of (1) by discretizing the white noise $\dot{W}$. First we introduce a discretization for the white noise. Let $\{\mathcal{T}_h\}$ be a family of triangulations of $\overline{\Omega}$ (see [1] for the requirements on $\{\mathcal{T}_h\}$), where $h \in (0, 1)$ is the meshsize. We assume that the family is quasiuniform, i.e., there exist positive constants ρ_1 and ρ_2 such that

$$\rho_1 h \leq R_T^{\text{inr}} < R_T^{\text{cir}} \leq \rho_2 h, \quad \forall\, T \in \mathcal{T}_h, \quad \forall\, 0 < h < 1, \tag{16}$$

where R_T^{inr} and R_T^{cir} are the inradius and the circumradius of T. Write

$$\xi_T^j = \frac{1}{\sqrt{|T|}} \int_T 1 \, dW^j(x) \quad j = 1, \cdots, d$$

for each triangle $T \in \mathcal{T}_h$, where $|T|$ denotes the area of T. It is well-known that $\{\xi_T^j\}_{T \in \mathcal{T}_h}$ is a family of independent identically distributed normal random variables with mean 0 and variance 1 (see [10]). Then the piecewise constant approximation to $\dot{W}^j(x)$ is given by

$$\dot{W}_h^j(x) = \sum_{T \in \mathcal{T}_h} |T|^{-\frac{1}{2}} \xi_T^j \chi_T(x) \tag{17}$$

where χ_T is the characteristic function of T. It is apparent that $\dot{W}_h = (W_h^1, \cdots, W_h^d) \in (L^d(\Omega))^2$ almost surely. However, we have the following estimate which shows that $\|\dot{W}_h\|$ is unbounded as $h \to 0$.

Lemma 2 [2] *Let $\|W_h\| = \sqrt{\|W_h^1\|^2 + \cdots + \|W_h^d\|^2}$. Then there exist positive constants C_1 and C_2 independent of h such that*

$$C_1 h^{-2} \leq E\left(\|\dot{W}_h\|^2\right) \leq C_2 h^{-2} \tag{18}$$

for $d = 2$ and

$$C_1 h^{-3} \leq E\left(\|\dot{W}_h\|^2\right) \leq C_2 h^{-3} \tag{19}$$

for $d = 3$.

Now we consider the approximation problem for (1) with the discretized white noise forcing term $\dot{W}_h$:

$$\begin{cases} -\nu \Delta u_h + \nabla p_h = f + \dot{W}_h, & \text{in } \Omega, \\ \operatorname{div} u_h = 0, & \text{in } \Omega, \\ u_h = 0, & \text{on } \partial\Omega. \end{cases} \tag{20}$$

Similar to (6) and (7), we define the weak solution for (20) as follows.

$$u_h(x) = \int_\Omega G(x, y) f(y) dy + \int_\Omega G(x, y) dW_h(y) \tag{21}$$

$$p_h(x) = \int_\Omega H(x, y) f(y)dy + \int_\Omega H(x, y)dW_h(y). \tag{22}$$

We have the following estimates concerning the bounds for u_h and p_h.

Lemma 3 *There exists a positive constant C independent of h such that*

$$E\left(\|u_h\|_2^2 + \|p_h\|_1^2\right) \leq Ch^{-2}. \tag{23}$$

Proof From the standard estimates of Stokes equation (see, e.g., [9]), we have that

$$\|u_h\|_2^2 + \|p_h\|_1^2 \leq C(\|f\|^2 + \|\dot{W}_h\|^2).$$

The result of the lemma is then a direct consequence of Lemma 2 and the above estimate. □

We have the following estimate for the errors $u - u_h$ and $p - p_h$.

Theorem 1 *Let (u, p) and (u_h, p_h) be the solution of (1) and (20) respectively. Then there exists a constant C such that*

$$E\left(\|u - u_h\|^2 + \|p - p_h\|_{-1}^2\right) \leq C|\ln h|h^2 \tag{24}$$

for $d = 2$ and

$$E\left(\|u - u_h\|^2 + \|p - p_h\|_{-1}^2\right) \leq Ch \tag{25}$$

for $d = 3$.

Proof We only prove (24). The proof of (25) is similar. Using Ito's isometry and Hölder inequality, we have that

$$\begin{aligned}
&E\left(\|u - u_h\|\right)^2 \\
&= E\left(\int_\Omega \left|\int_\Omega G(x, y)dW(y) - \int_\Omega G(x, y)dW_h(y)\right|^2 dx\right) \\
&= E\left(\int_\Omega \left|\sum_{T\in\mathcal{T}_h}\int_T G(x, y)dW(y) - |T|^{-1}\sum_{T\in\mathcal{T}_h}\int_T G(x, z)dz\int_T 1dW(y)\right|^2 dx\right) \\
&= E\left(\int_\Omega \left|\sum_{T\in\mathcal{T}_h}\int_T\int_T |T|^{-1}(G(x, y) - G(x, z))dzdW(y)\right|^2 dx\right) \\
&= \int_\Omega \left(\sum_{T\in\mathcal{T}_h}\int_T \left||T|^{-1}\int_T (G(x, y) - G(x, z))dz\right|^2 dy\right) dx.
\end{aligned}$$

From the Hölder inequality and (8) we obtain

$$\begin{aligned} E\left(\|u-u_h\|\right)^2 &\le \int_\Omega \left(\sum_{T\in\mathcal{T}_h} |T|^{-1} \int_T \int_T |G(x,y)-G(x,z)|^2 dz dy \right) dx \\ &= \sum_{T\in\mathcal{T}_h} |T|^{-1} \int_T \int_T \int_\Omega |G(x,y)-G(x,z)|^2 dx dz dy \\ &\le \sum_{T\in\mathcal{T}_h} |T|^{-1} \int_T \int_T C \ln|y-z||y-z|^2 dz dy \\ &\le C|\Omega||\ln h|h^2. \end{aligned}$$

Next we estimate $E\|p-p_h\|_{-1}^2$. Let $\phi \in H_0^1(\Omega)$. Using integration by parts, we have that

$$\begin{aligned} < p-p_h, \phi >^2 &= \left(\int_\Omega \left(\int_\Omega D(x,y)dW(y) - \int_\Omega D(x,y)dW_h(y) \right) \nabla\phi(x)dx \right)^2 \\ &\le \int_\Omega \left| \int_\Omega D(x,y)dW(y) - \int_\Omega D(x,y)dW_h(y) \right|^2 dx \|\phi\|_1^2. \end{aligned}$$

Thus

$$\|p-p_h\|_{-1}^2 \le \left(\int_\Omega \left| \int_\Omega D(x,y)dW(y) - \int_\Omega D(x,y)dW_h(y) \right|^2 dx \right).$$

Following the same procedure as in the estimate of $E\left(\|u-u_h\|\right)^2$, we obtain

$$E\left(\|p-p_h\|_{-1}\right)^2 \le |\ln h|h^2.$$

The proof of (24) is complete. □

3 Finite element approximations

3.1 Error estimate for the velocity u

In this section we consider the numerical approximations of (20) using the finite element method. Let $X := (H_0^1(\Omega))^d$ and $Q := L_0^2(\Omega) = \{q \in L^2(\Omega),\ \int_\Omega q(x)dx = 0\}$. We first define two bilinear forms as follows.

$$a(v,w) = \nu(\nabla v, \nabla w) \qquad \forall\, v, w \in X \tag{26}$$

and

$$b(v,q) = -(\mathrm{div} v, q) \qquad \forall\, X \in X,\ q \in Q \tag{27}$$

With a and b we define the variational formulation of (20) as

$$\begin{cases} a(u_h, v) + b(v, p_h) = (f, v) + (\dot{W}_h, v), & \forall v \in X, \\ b(u_h, q) = 0, \quad \forall q \in Q. \end{cases} \tag{28}$$

Next, we introduce finite dimensional subspaces $X_h \subset X$ and $Q^h \subset Q$ which are div-stable:

$$\inf_{0\neq q_h\in Q_h} \sup_{0\neq v_h\in X_h} \frac{b(v_h, q_h)}{\|v_h\|_1\|q_h\|_0} > \beta > 0 \tag{29}$$

where β is a constant independent of h, and satisfy the following approximation properties.

$$\begin{cases} \inf_{v_h\in X_h} \|v - v_h\|_s \leq Ch^{2-s}\|v\|_2 \quad s = 0, 1, \quad \forall\, v \in H^2(\Omega)^d, \\ \inf_{q\in Q_h} \|q - q_h\| \leq Ch\|q\|_1 \quad \forall\, q \in H^1(\Omega). \end{cases} \tag{30}$$

We refer to [5] for concrete constructions of X_h and M_h.

The finite element approximation for (20) is to find $\widehat{u}_h$ in X_h and $\widehat{p}_h$ in Q_h such that

$$\begin{cases} a(\widehat{u}_h, v_h) + b(v, \widehat{p}_h) = (f, v_h) + (\dot{W}_h, v_h), \quad \forall v_h \in X_h, \\ b(\widehat{u}_h, q_h) = 0, \quad \forall q_h \in Q_h. \end{cases} \tag{31}$$

We have the following error estimate for $u - \widehat{u}_h$.

Theorem 2 *There exists a constant C such that*

$$E(\|u - \widehat{u}_h\|^2) \leq C|\ln h|h^2 \tag{32}$$

for $d = 2$ and

$$E(\|u - \widehat{u}_h\|^2) \leq Ch \tag{33}$$

for $d = 3$.

Proof We only prove (32). The proof of (33) is similar. From the L^2 error estimate for the finite element solution $\widehat{u}_h$ [9], we have that

$$\|u_h - \widehat{u}_h\|^2 \leq Ch^4(\|u_h\|_2^2 + \|p_h\|_1^2).$$

By Lemma 3, there exists a constant C such that

$$E(\|u_h\|_2^2 + \|p_h\|_1^2) \leq Ch^{-2}.$$

The result (32) is then a direct consequence of the above estimates, (24), and the triangle inequality. □

Remark 1 As indicated in the proof of Theorem 3, the convergence rate for the finite element approximations of the deterministic Stokes equation problem is $O(h^2)$. The results of Theorem 3 demonstrate that the convergence rates of the finite element approximations of the stochastic Stokes equation problem are much lower.

Remark 2 It is well known that for a given random variable X, the convergence rate for the evaluation of EX using the Monte Carlo simulation is

$O\left(\frac{SD(X)}{\sqrt{M}}\right)$ where $SD(X)$ stands for the standard deviation of X and M is the sample size. Our finite element error estimates in Theorem 3 provide a guidance for the sample size needed in the Monte Carlo simulation of the statistics of the velocity field u. For instance, when $d = 2$, the sample size should be $O\left(\frac{1}{h^2 \ln^2 h}\right)$ so that the the Monte Carlo error matches the finite element error.

3.2 Error estimates for the pressure p

In this subsection we shall assume that the boundary $\partial\Omega$ of the convex set Ω belongs to $C^{2,1}$ (see page 4, [9] for a definition of $C^{2,1}$). Next we estimate the error between p and $\widehat{p}_h$ in H^{-1} norm. To this end we need the H^{-1} norm estimate for the pressure of deterministic Stokes equation. First we need the following result.

Lemma 4 *Let $g \in H^1(\Omega) \bigcap L_0^2(\Omega)$. There exists a constant C such that the solution (u, p) of the following problem*

$$\begin{aligned} -\nu\Delta u + \nabla p &= 0, \quad \text{in } \Omega, \\ \text{div} u &= g, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega \end{aligned} \tag{34}$$

has the following estimate:

$$\|u\|_2 + \|p\|_1 \leq C\|g\|_1.$$

Proof Let v_0 satisfy

$$\begin{aligned} \Delta v_0 &= g, \quad \text{in } \Omega, \\ v_0 &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

Define $u_0 = \text{grad} v_0$. Then $\text{div} u_0 = g$. Let $v = u - u_0$. Then (34) is equivalent to

$$\begin{aligned} -\nu\Delta v + \nabla q &= \nu\Delta u_0, \quad \text{in } \Omega, \\ \text{div} v &= 0, \quad \text{in } \Omega, \\ v &= u_0, \quad \text{on } \partial\Omega. \end{aligned} \tag{35}$$

From the standard theory of Stokes equation [9], we know that (35) has a unique solution and there exists a constant C such that

$$\begin{aligned} \|v\|_2 + \|q\|_1 &\leq C(\|u_0\|_{H^{\frac{3}{2}}(\partial\Omega)} + \|\Delta u_0\|) \\ &\leq C(\|v_0\|_{H^{\frac{5}{2}}(\Omega)} + \|v_0\|_3) \\ &\leq C\|v_0\|_3 \leq C\|g\|_1 \end{aligned}$$

which proves the lemma. □

Let (w_h, q_h) be the finite element approximations for the solution (w, q) of the following deterministic Stokes equation:

$$\begin{aligned} -\nu\Delta w + \nabla q = l, &\quad \text{in } \Omega, \\ \text{div} w = 0, &\quad \text{in } \Omega, \\ w = 0, &\quad \text{on } \partial\Omega \end{aligned} \tag{36}$$

where $l \in L^2(\Omega)$. We have the following error estimate for the pressure q in H^{-1} norm.

Proposition 1 *There exists a constant C such that*

$$\|q - q_h\|_{-1} \leq Ch^2 \left(\|w\|_2 + \|q\|_1\right). \tag{37}$$

Proof First notice that the variational forms for (w, q) and (w_h, q_h) are

$$\begin{cases} a(w, v) + b(v, q) = (l, v), & \forall v \in X, \\ b(w, \mu) = 0, & \forall \mu \in Q \end{cases} \tag{38}$$

and

$$\begin{cases} a(w_h, v_h) + b(v_h, q_h) = (l, v_h), & \forall v_h \in X_h, \\ b(w_h, \mu) = 0, & \forall \mu \in Q_h, \end{cases} \tag{39}$$

respectively. Let $g \in H^1(\Omega) \bigcap L_0^2(\Omega)$ and consider the following problem: find $\phi_g \in X$ and $\psi_g \in Q$ such that

$$\begin{cases} a(v, \phi_g) + b(v, \psi_g) = 0, & \forall v \in X, \\ b(\phi_g, \mu) = (\mu, g), & \forall \mu \in Q. \end{cases} \tag{40}$$

From the above three equations we have that

$$\begin{aligned} (q - q_h, g) &= b(\phi_g, q - q_h) \\ &= b(\phi_g - v_h, q - q_h) + b(v_h, q - q_h) \\ &= b(\phi_g - v_h, q - q_h) + a(w, v_h) - a(w_h, v_h) \\ &= b(\phi_g - v_h, q - q_h) + a(w - w_h, v_h) - a(w - w_h, \phi_g) \\ &\quad -b(w - w_h, \psi_g) \\ &= b(\phi_g - v_h, q - q_h) + a(w - w_h, v_h - \phi_g) - b(w - w_h, \psi_g - \mu_h), \\ &\quad \forall v_h \in X_h, \quad \forall \mu_h \in Q_h. \end{aligned}$$

Thus

$$\begin{aligned} |(q - q_h, g)| &\leq C(\|\phi_g - v_h\|_1 \|q - q_h\| + \|w - w_h\|_1 \|\phi_g - v_h\|_1 \\ &\qquad + \|w - w_h\|_1 \|\psi_g - \mu_h\|) \\ &\leq (\|w - w_h\|_1 + \|q - q_h\|)(\|\phi_g - v_h\|_1 + \|\psi_g - \mu_h\|). \end{aligned}$$

By the error estimates for Stokes equations [9], we have

$$\|w - w_h\|_1 + \|q - q_h\| \le Ch(\|w\|_2 + \|q\|_1).$$

Thus

$$\begin{aligned}|(q - q_h, g)| &\le Ch(\|w\|_2 + \|q\|_1)(\inf_{v_h \in X_h} \|\phi_g - v_h\|_1 + \inf_{\mu_h \in Q_h} \|\psi_g - \mu_h\|) \\ &\le Ch^2((\|w\|_2 + \|q\|_1)(\|\phi_g\|_2 + \|\psi_g\|_1).\end{aligned}$$

From Lemma 4 we have that

$$\frac{|(q - q_h, g)|}{\|g\|_1} \le Ch^2(\|w\|_2 + \|q\|_1). \tag{41}$$

Since $g \in H_0^1$ is arbitrary in (41), we have that

$$\|q - q_h\|_{-1} \le Ch^2\left(\|w\|_2 + \|q\|_1\right).$$

The proof is complete. □

Similar to Theorem 2, we have the following error estimate for $p - p_h$ in H^{-1} norm.

Theorem 3 *There exists a constant C independent of h such that*

$$E(\|p - \widehat{p}_h\|_{-1}^2) \le C|\ln h|h^2$$

for $d = 2$ *and*

$$E(\|p - \widehat{p}_h\|_{-1}^2) \le Ch$$

for $d = 3$.

Proof The result of the theorem is a direct consequence of Theorem 1, Lemma 3 and Proposition 1. □

Remark 3 All the analysis and error estimates are valid when the divergence free equation $\mathrm{div} u = 0$ in (1) is replaced by $\mathrm{div} u = g$ where g is a function defined on Ω with sufficient regularity (see [9]).

4 Numerical experiments

In this section we will present a numerical experiment using the finite element method described in Sections 2 and 3. We construct the finite dimensional subspaces X_h and M_h using the Taylor-Hood method (see page 177, [9]). The numerical algorithm consists of three steps.

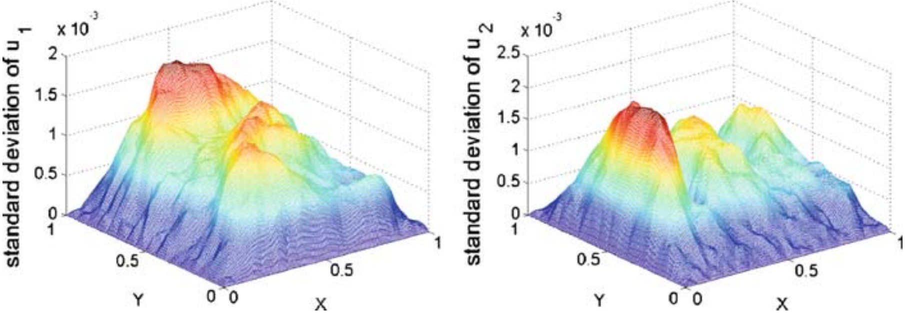

Fig. 1 Approximate standard deviation of $u = (u_1, u_2)$: h = 1/16, sample size = 10000

Step 1 For $m = 1, \cdots, M$, generate samples $\dot{W}_h^m = \sum_{T \in \mathcal{T}_h} |T|^{-\frac{1}{2}} \xi_T^m \chi_T(x)$, of the discretized white noise $\dot{W}_h$ by generating samples $\{\xi_T^m\}_{T \in \mathcal{T}_h}$ of $\{\xi_T\}_{T \in \mathcal{T}_h}$ where M is the sample size;

Step 2 For $m = 1, \cdots, M$, Solve (20), with $\dot{W}_h$ replaced by $\dot{W}_h^m$, to obtain the approximate solutions $(\widehat{u}_h^m, \widehat{p_h}^m)$ of $(\widehat{u}_h, \widehat{p_h})$ by the finite element method;

Step 3 Evaluate statistics $E(v(\widehat{u}_h, \widehat{p}_h))$ (for example, if $v(u, p) = |u|^2$, then the statistics is the second moment of u) using the Monte Carlo method:

$$E(v(\widehat{u}_h, \widehat{p_h})) \approx \frac{1}{M} \sum_{m=1}^{M} v(\widehat{u}_h^m, \widehat{p_h}^m).$$

In the numerical experiments, we consider the stochastic Stokes problem with $\Omega = [0, 1] \times [0, 1]$, $d = 2$, $f = 2\pi^2 \sin \pi x \sin \pi y$ and $\operatorname{div} u = \pi \sin \pi(x + y)$ (see Remark 3). Since $E(\dot{W}) = 0$, it is easy to check that $(Eu, Ep) = (\sin \pi x \sin \pi y, \sin \pi x \sin \pi y, 0)$ is the exact solution of the deterministic Stokes problem, i.e., (Eu, Ep) satisfy

$$\begin{cases} -\nu \Delta Eu + \nabla Ep = f, & \text{in } \Omega, \\ \operatorname{div} Eu = \pi \sin \pi(x + y), & \text{in } \Omega, \\ Eu = 0, & \text{on } \partial\Omega. \end{cases}$$

We first discover, as shown in Fig. 1, that the variance/standard deviation of the velocity $u = (u_1, u_2)$ is quite small (in the order of 10^{-3}), which indicates that we can perform Monte Carlo simulations with relatively small sample sizes (see Remark 2).

Table 1 Convergence analysis of $E\widehat{u}_h$

h	M	$\|Eu_1 - E(u_1)_h\|$	Conv.rate	$\|Eu_2 - E(u_2)_h\|$	Conv.rate
0.25000	16	1.7483E-3	–	1.8011E-3	–
0.12500	256	4.5359E-4	1.9465	4.0975E-4	2.1361
0.06250	4096	1.3668E-4	1.7306	1.0567E-4	1.9552
0.03125	65536	3.5957E-5	1.9265	2.6948E-5	1.9713

Table 2 Convergence analysis of $E\|\widehat{u}_h\|^2$

h	M	$E\|\widehat{u}_h\|^2$	$\|E\|\widehat{u}_h\|^2 - \|E\|\widehat{u}_{\frac{h}{2}}\|^2\|$	Conv.rate
0.25000	16	0.49978993	3.4397e-4	–
0.12500	64	0.50013390	1.6948e-4	1.0212
0.06250	256	0.49996444	7.2205e-5	1.2309
0.03125	1024	0.50003663	–	–

In Table 1, we list the errors and convergence rates of the expectation $E\widehat{u}_h$ of velocity $\widehat{u}_h = (u_h^1, u_h^2)$. Notice that since $E\dot{W}_h = 0$, $E\widehat{u}_h$ can be evaluated as the solution of the deterministic finite element approximation. We use the Monte Carlo simulations here in order to verify that sufficient numbers of samples have been used. Since the convergence rate of $E\widehat{u}_h$ to Eu is $O(h^2)$ (see Remark 1), the increase of the sample size is increased by 16 folds as the meshsize is decreased by half. (see Remark 2).

In Table 2, we list the results of numerical approximations of the second moments of u. Since the exact solution of $E\|u\|^2$ is not known, we use $|E\|\widehat{u}_h\|^2 - E\|\widehat{u}_{\frac{h}{2}}\|^2|$ to estimate the errors. Because of the slower convergence rate for the stochastic problem, we only need to increase the sample size by 4 folds as the meshsize is decreased by half.

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References

1. Brenner, S., Scott, L.: The Mathematical Theory of Finite Element Method. Springer, New York (1994)
2. Cao, Y., Yang, H., Yin, L.: Finite element methods for semilinear elliptic stochastic partial differential equations. Numer. Math. **106**, 181–198 (2007)
3. Weinan, E., Mattingly, J., Sinai, Y.: Gibbsian dynamics and ergodicity for the stochastic forced Navier-Stokes equation. Comm. Math. Phys. **224**(1), 83–106 (2001)
4. Flandoli, F.: Dissipativity and invariant measures for stochastic Navier-Stokes equations. NoDEA **1**, 403–426 (1994)
5. Gunzburger, M.: Finite Element Methods for Viscous Incompressible Flows: A Guide to Theory, Practice, and Algorithms. Academic, London (1989)
6. Hou, T., Luo, W., Rozovskii, B., Zhou, H.: Wiener chaos expansions and numerical solutions of randomly forced equations of fluid mechanics. J. Comput. Phys. **216**, 687–706 (2006)
7. Ladyzhenskaya, O.: The Mathematical Theory of Viscous Incompressible Flows, 2nd edn. Gordon and Breach, New York (1969)
8. Mattingly, J., Pardoux, E.: Malliavin calculus for the stochastic 2d Navier-Stokes equation. Comm. Pure Appl. Math. **59**, 1742–1790 (2006)
9. Girault, V., Raviart, P.: Finite Element Methods for Navier-Stokes Equations. Springer, Berlin (1986)
10. Walsh, J.: An introduction to stochastic partial differential equations. In: Lecture Notes in Mathematics 1180, pp. 265–439. Springer, New York (1986)
11. Xiu, D., Karniadakis, G.: Modeling uncertainty in flow simulations via generalized polynomial chaos. J. Comput. Phys. **187**, 137–167 (2003)