RIESZ BASES, MULTIRESOLUTION ANALYSES, AND PERTURBATION

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Abstract. This article is partly a review of known results and partly a research paper. We study affine sequences with dilation factor 2, as well as sequences of translates, generated by a single function in $L^2(\mathbb{R})$.

In the first section we introduce our notation, some definitions, and some basic theorems. In Section 2 we discuss properties of frame and Riesz bases of translates and clarify the proofs of several theorems. Section 3 reviews results on perturbations of the Haar wavelet. In Section 4 we study the characterization of MRA orthonormal and Riesz wavelets. We also discuss the relationship between functions associated with an MRA, and functions generated by an MRA. Finally, Section 5 contains old and new results on the convolution of Riesz bases, scaling functions, and multiresolution analyses.

Many of the results discussed here have been obtained for the univariate case only. Their extension to a more general setting is a source of interesting open problems.

1. Introduction

In this section we will introduce our notation, some definitions, and some basic theorems.

In what follows $\mathbb{Z}$ will denote the integers, $\mathbb{Z}^+$ the nonnegative integers, and $\mathbb{R}$ the real numbers; t and x will always denote real variables. The Fourier transform of a function f will be denoted by $\widehat{f}$. If $f \in L(\mathbb{R})$,

$$\widehat{f}(x) := \int_{\mathbb{R}} e^{-2\pi txi} f(t)\, dt.$$

Other definitions of the Fourier transform are common in the literature. Since all these definitions are equivalent modulo a linear change of variable and, in some cases, multiplication by a suitable constant, we have modified, where needed, all the statements quoted herein, so as to conform to the definition given above.

Let $\mathbb{H}$ be a (separable) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \| := \langle \cdot, \cdot \rangle^{1/2}$, let $\Lambda \subset \mathbb{Z}$, and let $\mathbf{F} := \{f_k, k \in \Lambda\} \subset \mathbb{H}$. $\mathbf{F}$ is called a *frame* if there are constants $0 < A \leq B$ such that for every $f \in \mathbb{H}$

$$A\|f\|^2 \leq \sum_{k \in \Lambda} |\langle f, f_k \rangle|^2 \leq B\|f\|^2. \tag{1.1}$$

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The constants A and B are called (lower and upper) *bounds* of the frame. The operator $S : \mathbb{H} \to \mathbb{H}$ defined by

$$Sf := \sum_{k \in \Lambda} \langle f, f_k \rangle f_k$$

is called the *frame operator*. It is invertible and self–adjoint, and for every $f \in \mathbb{H}$

$$f = \sum_{k \in \Lambda} \langle f, S^{-1} f_k \rangle f_k = \sum_{k \in \Lambda} \langle f, f_k \rangle S^{-1} f_k.$$

If only the right-hand inequality in (1.1) is satisfied for all $f \in \mathbb{H}$, then $\mathbf{F}$ is called a *Bessel sequence* with bound B. If $\mathbf{a} := \{a_k; k \in \Lambda\}$, then $\mathbf{F}$ is a Bessel sequence with Bessel bound B if and only if the *analysis operator*

$$T : \mathbf{a} \longrightarrow \sum_{k \in \Lambda} a_k f_k$$

is a well defined bounded operator from $\ell^2(\Lambda)$ into $\mathbb{H}$ and $||T|| \leq \sqrt{B}$. T is also called the *pre–frame operator*

$\mathbf{F}$ is called a *Riesz basis* if its linear span is dense in $\mathbb{H}$ and there are constants A and B, $A > 0$, such that for every sequence $\mathbf{a} \in \ell^2(\Lambda)$

$$\sqrt{A}||\mathbf{a}|| \leq ||T\mathbf{a}|| \leq \sqrt{B}||\mathbf{a}||. \tag{1.2}$$

The constants A and B are called (lower and upper) *bounds* of the Riesz basis. Every orthonormal basis is a Riesz basis with bounds $A = B = 1$, every Riesz basis is a frame, and Riesz bounds and frame bounds coincide. A sequence is a Riesz basis if and only if it is a frame having the additional property that upon the removal of any element from the sequence, it ceases to be a frame.

$\mathbf{F}$ is called a *frame sequence* (resp. a *Riesz sequence*) if it is a frame (resp. a Riesz basis) in the closure of its linear span. Riesz sequences are also called *exact frame sequences.* Note that if $\mathbf{F}$ is an orthogonal sequence, then it is an orthogonal basis of the closure of its linear span.

Two sequences $\mathbf{F} := \{f_k, k \in \Lambda\}$ and $\mathbf{G} := \{g_k, k \in \Lambda\}$ in $\mathbb{H}$ are said to be *biorthogonal* if $\langle f_m, g_k \rangle = \delta_{m,k}$ for every m and k; we also say that $\mathbf{G}$ is biorthogonal to $\mathbf{F}$. If $\mathbf{F}$ is a Riesz basis then it has a unique biorthogonal sequence $\mathbf{G}$, and this sequence is a Riesz basis. Moreover, if A and B are Riesz bounds for $\mathbf{F}$, then $1/B$ and $1/A$ are Riesz bounds for $\mathbf{G}$. For a discussion of the theory of frames and Riesz bases see [14], and also [7, 43].

Here is a characterization of frame and Riesz sequences that will be useful in the subsequent discussion:

Theorem 1.1.
(a) *A sequence* $\mathbf{F} \in \mathbb{H}$ *is a frame sequence if and only if the pre-frame operator* T *satisfies (1.2) for every* $\mathbf{a} \in (\ker T)^{\perp}$.
(b) *In particular,* $\mathbf{F}$ *is a Riesz sequence if and only if* T *is one–to–one.*

The proof of (a) follows from [14, Lemma 5.5.4]. Since T is one–to–one if and only if $\ker T = \{0\}$, (b) follows trivially from the definition of Riesz sequence.

In this paper the underlying Hilbert space will be $L^2(\mathbb{R})$ with the usual inner product and norm, and we will study affine sequences with dilation factor 2 generated by a single function, i.e. sequences of the form $\{\psi_{j,k}; j, k \in \mathbb{Z}\}$, where

$\psi \in L^2(\mathbb{R})$ and $\psi_{j,k}(t) := 2^{j/2}\psi(2^j t - k)$, as well as sequences of translates generated by a single function, i.e., sequences of the form $\{\varphi_k; k \in \mathbb{Z}\}$, where $\varphi \in L^2(\mathbb{R})$ and $\varphi_k := \varphi_{0,k} = \varphi(t-k)$.

Although the rest of the theorems in this article are stated for functions of a real variable, we direct the reader to generalizations to the multivariate case, whenever we are aware of any.

A function $\psi \in L^2(\mathbb{R})$ will be called a frame wavelet, a Bessel wavelet, a Riesz wavelet, or an orthonormal wavelet, if the affine sequence $\{\psi_{j,k}; j,k \in \mathbb{Z}\}$ it generates is respectively a frame, a Bessel sequence, a Riesz basis, or an orthonormal basis of $L^2(\mathbb{R})$. When we refer to the bounds of a wavelet ψ we mean the frame, Bessel, or Riesz bounds, as the case may be, of the affine sequence generated by ψ. A Riesz wavelet ψ is said to be biorthogonal if there is a function $\tau \in L^2(\mathbb{R})$ such that $\{\tau_{j,k}; k \in \mathbb{Z}\}$ is biorthogonal to $\{\psi_{j,k}; k \in \mathbb{Z}\}$. The discussion of the preceding paragraph implies that τ is a Riesz wavelet. It is also easy to see that τ must be unique. We call τ the (Riesz) wavelet biorthogonal to ψ.

A *multiresolution analysis* (MRA) is a sequence $\{V_j; j \in \mathbb{Z}\}$ of closed linear subspaces of $L^2(\mathbb{R})$ such that:

(i)
$$V_j \subset V_{j+1} \text{ for every } j \in \mathbb{Z}.$$

(ii)
$$\text{For every } j \in \mathbb{Z}, f(t) \in V_j \text{ if and only if } f(2t) \in V_{j+1}.$$

(iii)
$$\bigcup_{j\in\mathbb{Z}} V_j \text{ is dense in } L^2(\mathbb{R}).$$

(iv)
$$\text{There is a function } \varphi \text{ such that } \{\varphi_k; k \in \mathbb{Z}\} \text{ is an orthonormal basis of } V_0.$$

For any function $\varphi \in L^2(\mathbb{R})$, let
$$\Phi(x) := \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(x+k)|^2.$$
If $\{\varphi_k; k \in \mathbb{Z}\}$ is a Riesz basis of V_0, and $\sigma(x) := \widehat{\varphi}(x)[\Phi(x)]^{-1/2}$, it is well known that $\{\sigma_k; k \in \mathbb{Z}\}$ is an orthonormal basis of V_0 (see, e.g., [17, p. 139], and also Theorem 2.2 below). Thus, condition (iv) can be replaced by the following:

(iv′)
$$\text{There is a function } \varphi \text{ such that } \{\varphi_k; k \in \mathbb{Z}\} \text{ is a Riesz basis of } V_0.$$

Traditionally, the definition of MRA includes an additional condition:

(v)
$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}.$$
It was later discovered that (iv′) (and therefore (iv)) implies (v) [17, p. 141, Proposition 5.3.1] (see also [28, p. 45, Theorem 1.6] for a proof that (i), (ii), and (iv) imply (v)).

It follows from the definition of MRA that there is a 1–periodic function $p \in L^2[0,1]$ such that

$$\widehat{\varphi}(2x) = p(x)\widehat{\varphi}(x) \qquad \text{a.e.} \tag{1.3}$$

The function φ is called a *scaling function* for the MRA, (1.3) is called a *two scale equation*, and p is called the *low pass filter* associated with φ. If $\{\varphi_k; k \in \mathbb{Z}\}$ is an orthonormal basis of V_0, then φ will be called an *orthonormal scaling function* for the MRA. This definition is consistent with [28, p. 53]. For a thorough study of scaling functions see [38].

A function will be called a (orthonormal) scaling function, if it is a (orthonormal) scaling function for some MRA.

The following theorem shows that in certain cases, condition (iii) may be replaced by the assumption that the function φ does not vanish at the origin. It will be used in the proof of Theorem 5.8 below.

Theorem 1.2. [28, p. 46, Theorem 1.7]. *Let $\{V_j; j \in \mathbb{Z}\}$ be a sequence of closed linear subspaces of $L^2(\mathbb{R})$ satisfying (i), (ii), and (iv); assume that the scaling function φ of condition (iv) is such that $|\widehat{\varphi}|$ is continuous at 0. Then (iii) is satisfied if and only if $\widehat{\varphi}(0) \neq 0$. If either is the case, $|\widehat{\varphi}(0)| = 1$.*

The basic characterization of orthonormal wavelets is due to Gripenberg [24] and Wang [41]. See the comments at the end of Chapter 7 of [28] for an account of the history of this result.

Theorem 1.3. [28, p. 332]. *A function $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet if and only if the following three conditions are satisfied:*

(a) $\|\psi\| = 1$.

(b) $\displaystyle\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j x)|^2 = 1 \qquad a.e.$

(c) $\displaystyle\sum_{j\in\mathbb{Z}^+} \widehat{\psi}(2^j x)\, \overline{\widehat{\psi}(2^j(x+m))} = 0, \qquad a.e., \qquad m \in 2\mathbb{Z}+1.$

Theorem 1.3 has been extended to several variables by M. Frazier, G. Garrigós, K. Wang, and G. Weiss [19] (see also [18]). This theorem allows us to determine whether a given function is an orthonormal wavelet, but it does not tell us how to actually construct a wavelet. This problem, in its full generality, is still open. A constructive method for MRA orthogonal wavelets will be discussed in Section 4.

2. Frames and Riesz bases of translates

In what follows we will use the following notation: $I := [0,1]$, and, for $\varphi \in L^2(\mathbb{R})$,

$$r(x) := [\Phi(x)]^{1/2} = \left[\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(x+k)|^2\right]^{1/2}. \tag{2.1}$$

If $\Lambda \subset \mathbb{Z}$, define

$$H_\Lambda := \overline{\text{span}}\,\{e^{2\pi k x i}; k \in \Lambda\},$$

and let E_Λ be the closed subspace of H_Λ of all f such that $f(x)\Phi(-x) = 0$ a.e.

The following is part of the statement of [10, Theorem 2.1]. The proof is essentially the original one.

Theorem 2.1. *Suppose* $\varphi \in L^2(\mathbb{R})$, *let* $\Lambda \subset \mathbb{Z}$, *and let* $\mathbf{S} := \{\varphi_k; k \in \Lambda\}$. *Then:*
(a) $\mathbf{S}$ *is a frame sequence with frame bounds* A *and* B *if and only if for every* $f \in H_\Lambda \cap E_\Lambda^\perp$,

$$A\,||f||^2 \le \int_I |f(x)|^2 \Phi(-x)\,dx \le B\,||f||^2. \tag{2.2}$$

(b) *Furthermore, if the condition of (a) is satisfied, then* $\mathbf{S}$ *is a Riesz sequence with the same bounds if and only if* $E_\Lambda = \{0\}$.

Proof. (a) Note that $\mathbf{a} := \{a_k; k \in \Lambda\} \in \ell^2(\Lambda)$ if and only if

$$f(x) := \sum_{k\in\Lambda} a_k e^{2\pi kxi} \in H_\Lambda.$$

If T denotes the analysis operator, then, using Plancherel's identity we have:

$$\begin{aligned}
||T\mathbf{a}||^2 &= \left\|\sum_{k\in\Lambda} a_k\varphi_k\right\|^2 = \int_{\mathbb{R}} \left|\sum_{k\in\Lambda} a_k e^{-2\pi kxi}\widehat{\varphi}(x)\right|^2 dx \\
&= \int_{\mathbb{R}} \left|\sum_{k\in\Lambda} a_k e^{2\pi kxi}\widehat{\varphi}(-x)\right|^2 dx \\
&= \int_I \left|\sum_{k\in\Lambda} a_k e^{2\pi kxi}\right|^2 \sum_{r\in\mathbb{Z}} |\widehat{\varphi}(-x-r)|^2\,dx \\
&= \int_I \left|\sum_{k\in\Lambda} a_k e^{2\pi kxi}\right|^2 \sum_{r\in\mathbb{Z}} |\widehat{\varphi}(-x+r)|^2\,dx = \int_I |f(x)|^2\Phi(-x)\,dx
\end{aligned}$$

This implies that $\mathbf{a} \in \ker T$ if and only if $f(x)\Phi(-x) = 0$ a.e., and the assertion follows from Theorem 1.1 (a).
(b) Recall first, that for a Riesz sequence frame bounds and Riesz bounds coincide. Since T is one–to–one if and only if $\ker T = \{0\}$ (here "0" denotes a vector in $\ell^2(\Lambda)$), and this condition in turn is satisfied if and only if $E_\Lambda = \{0\}$ (here "0" denotes a function), the assertion follows from Theorem 1.1 (b). □

The following result was originally proved by Benedetto and Li with a mild restriction [5, 7]. The more general version that we state here, was proved by Di–Rong Chen [11], Casazza, Christensen and Kalton [10], and Kim, Kim and Li [31] (see also [6]). It can be derived by arguments similar to those that were used to prove Theorem 2.1. For a detailed proof see [14, Theorem 7.2.3].

Theorem 2.2. *Let* $\varphi \in L^2(\mathbb{R})$ *and* $\mathbf{S} := \{\varphi_k; k \in \mathbb{Z}\}$. *For any* $A, B > 0$ *the following characterizations hold:*

(a) $\mathbf{S}$ *is a Bessel sequence with bound* B *if and only if*

$$\Phi(x) \le B \qquad \text{a.e. } x \in I.$$

(b) $\mathbf{S}$ *is an orthonormal sequence if and only if*

$$\Phi(x) = 1 \qquad \text{a.e. } x \in I.$$

(c) $\mathbf{S}$ *is a Riesz sequence with bounds* $A \le B$ *if and only if*

$$A \le \Phi(x) \le B \qquad \text{a.e. } x \in I.$$

(d) $\mathbf{S}$ *is a frame sequence with bounds* $A \le B$ *if and only if*

$$A \le \Phi(x) \le B \qquad \text{a.e. } x \in I \setminus N,$$

where $N := \{x \in I : \Phi(x) = 0\}$.

The proof of the next theorem relies on the properties of the Hardy class H^2. Let $p > 0$. A function $f(z)$ is said to *belong to the class* H^p if it is analytic in the unit disc $|z| < 1$, and the set of integrals

$$\int_0^{2\pi} |f(re^{xi})|^p \, dx$$

is bounded for $0 < r < 1$.

The case $p = 2$ is particularly simple, since if

$$f(z) := \sum_{k \in \mathbb{Z}^+} a_k z^k = \sum_{k \in \mathbb{Z}^+} a_k r^k e^{kxi}, \tag{2.3}$$

Bessel's identity

$$\int_0^{2\pi} |f(re^{xi})|^2 \, dx = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}^+} |a_k|^2 r^{2k}$$

shows that $f \in H^2$ if and only if $\{a_k; k \in \mathbb{Z}^+\} \in \ell^2(\mathbb{Z}^+)$ (cf., e.g., [46]).

Here is an interesting result of Casazza, Christensen and Kalton (it appears as a remark following the proof of [10, Theorem 2.4]). The argument we will use can be found there. We have added some details to make the proof more transparent. Note that if $\varphi(t) = 0$ a.e., then trivially $\{\varphi_k; k \in \Lambda\}$ is a frame, but not a Riesz basis.

Theorem 2.3. *Let* $\varphi \in L^2(\mathbb{R})$, *and assume that* $\varphi(t)$ *does not vanish a.e. Then* $\mathbf{S_0} := \{\varphi_k; k \in \mathbb{Z}^+\}$ *is a frame sequence if and only if it is a Riesz sequence.*

Proof. Assume $\mathbf{S_0}$ is a frame sequence with bounds $0 < A \le B$. From Theorem 2.1 (a) we know that (2.2) is satisfied for every $g \in H_{\mathbb{Z}^+} \cap E_{\mathbb{Z}^+}^{\perp}$. If $g \in H_{\mathbb{Z}^+}$, then

$$g(x) = \sum_{k \in \mathbb{Z}^+} a_k e^{2\pi k x i}, \qquad \{a_k : k \in \mathbb{Z}^+\} \in \ell^2(\mathbb{Z}^+).$$

Now, the function $f(z)$ defined by (2.3) is clearly in the Hardy class H^2. Thus, if in particular $g \ne 0$ (whence $f \ne 0$), we deduce from [29, p. 90] or [46, Vol 1., Chapter VII, Theorem 7.25] that $f(e^{xi}) \ne 0$ a.e. Since $g(x) = f(e^{2\pi xi})$ and $\Phi(-x)$ cannot vanish a.e. (for otherwise $\varphi(t)$ would vanish a.e.), we conclude that $E_{\mathbb{Z}^+} = \{0\}$, and the assertion follows from Theorem 2.1 (b). □

We also have:

Theorem 2.4. *Let* $\varphi \in L^2(\mathbb{R})$ *and* $\mathbf{S_0} := \{\varphi_k; k \in \mathbb{Z}^+\}$. *If* $\mathbf{S_0}$ *is a frame sequence then* $\mathbf{S} := \{\varphi_k; k \in \mathbb{Z}\}$ *is a frame sequence.*

Proof. Assume first that f is a trigonometric polynomial in $H_{\mathbb{Z}^+} \cap E_{\mathbb{Z}^+}^{\perp}$. Then Theorem 2.2 implies that f satisfies (2.2). Suppose now that f is any trigonometric polynomial. Then for large enough n we have that $e^{2\pi n x i} f \in H_{\mathbb{Z}^+}$. Thus (2.2) holds for all trigonometric polynomials in $H_{\mathbb{Z}} \cap E_{\mathbb{Z}}^{\perp}$, and therefore for all functions in $H_{\mathbb{Z}} \cap E_{\mathbb{Z}}^{\perp}$. Applying Theorem 2.2 (a) we conclude that $\mathbf{S}$ is a frame sequence. □

The preceding proof has been culled from [10, Theorem 2.4].

3. Perturbations of the Haar Wavelet

Let ψ be a frame wavelet in $L^2(\mathbb{R})$; for $j \in \mathbb{Z}$, let P_j denote the closure of the linear span of $\{\psi_{j,k}; k \in \mathbb{Z}\}$, and let $V_j := \sum_{r<j} P_r$. Note that $\psi \in V_1$. We say that ψ is *associated* with an MRA, or that ψ is an MRA wavelet, if $M := \{V_j; j \in \mathbb{Z}\}$ is a multiresolution analysis; we also say that ψ is associated with M. This definition is apparently due to Wang [41].

A feature of MRA orthonormal wavelets is the existence of the *fast wavelet transform algorithm*, which allows for the computation of the wavelet coefficients in $O(n)$ steps, where n is the sample size [34]. This compares favorably with the *fast Fourier transform algorithm*, which allows for the computation of the Fourier coefficients in $O(n \log n)$ steps [9]. If, in addition, the wavelet is of compact support and smooth, it has the additional advantages of having good localization in both time and frequency. Not everything is perfect, though, since most MRA orthonormal wavelets do not have a closed form representation and have to be obtained recursively using the cascade algorithm [17, 40]. Moreover, compactly supported MRA orthonormal wavelets with dilation factor 2 lack symmetry: Recall that the Haar function is defined by

$$\psi_H(t) := \begin{cases} 1 & \text{if } 0 \le t < 1/2 \\ -1 & \text{if } 1/2 \le t \le 1 \\ 0 & \text{otherwise} \end{cases} .$$

We have:

Theorem 3.1. [17, Theorem 8.1.4] *Suppose that φ and ψ, the scaling function and orthonormal wavelet (with dilation factor 2) associated with a multiresolution analysis, are both real and compactly supported. If ψ has either a symmetry or an antisymmetry axis, then ψ is the Haar function.*

In view of this theorem, we know that we cannot have an orthonormal wavelet with dilation factor 2 that is symmetric, smooth, and of compact support. We cannot have all four properties, but we can sacrifice orthonormality in order to obtain symmetry: Given $\varepsilon > 0$ and an arbitrary positive integer m, we can perturb the Haar wavelet into a Riesz wavelet $\psi_H^{\{\varepsilon,m\}} \in C^m(-\infty,\infty)$ with support in $[-\varepsilon, 1+\varepsilon]$ (thus having good time and frequency localization), and known Riesz bounds. This perturbation is based on the observation that adding a Bessel sequence with a small bound to a Riesz basis transforms the original basis into another Riesz basis. Moreover, Riesz bounds for the new basis can be expressed in terms of the Riesz bounds of the original basis:

Theorem 3.2. [20, Theorem 5]. *Let $\{f_n\}$ be a Riesz basis in a Hilbert space $\mathbb{H}$ with bounds A and B. Assume $\{g_n, n \in \mathbb{Z}\} \subset \mathbb{H}$ is such that $\{f_n - g_n, n \in \mathbb{Z}\}$ is a Bessel sequence with bound $M < A$. Then $\{g_n, n \in \mathbb{Z}\}$ is a Riesz basis with bounds $[1-(M/A)^{1/2}]^2 A$ and $[1+(M/B)^{1/2}]^2 B$. Conversely, if $\{f_n, n \in \mathbb{Z}\}$ and $\{g_n, n \in \mathbb{Z}\}$ are Riesz bases in $\mathbb{H}$ with bounds A_1, B_1 and A_2, B_2 respectively, and U is a linear homeomorphism such that $Uf_n = g_n, n \in \mathbb{Z}$, then $\{f_n - g_n, n \in \mathbb{Z}\}$ is a Bessel sequence with bound $M :=$ min $\{B_1 \|I-U\|^2, B_2 \|I-U^{-1}\|^2\}$.*

A similar theorem for frames was obtained by Christensen [12, 13]. A very nice discussion of frame and Riesz basis perturbations in Hilbert spaces can be found in Chapter 15 of [14].

We now describe how the perturbed Haar wavelets $\psi_H^{\{\varepsilon,m\}}$ are defined. The basic idea is to excise intervals around the discontinuities of the Haar function and replace them with functions based on B–splines of designated smoothness.

Let $N_m(t)$ denote the B–spline of order m ($m \geq 2$) [15, Chapter 4], $\chi_{[0,m-1]}(t)$ the characteristic function of $[0, m-1]$,

$$g_m(t) := \chi_{[0,m-1]}(t) \sum_{k=0}^{m-2} N_m(t-k),$$

$$g_{m,1}(t) := g_m(t-m+1), \quad h_m(t) := (1/2) \sum_{k=0}^{m-2} N_m(t-k),$$

and

$$q_m(t) := g_{m,1}(t) - h_m(t).$$

For $0 < \varepsilon < 1/2$, let

$$\begin{gathered} \alpha_{\varepsilon,m,1} := -\alpha_{\varepsilon,m,2} := -\alpha_{\varepsilon,m,3} := \alpha_{\varepsilon,m,4} := 2(m-1)/\varepsilon, \\ \beta_{\varepsilon,m,1} := 2(m-1), \qquad \beta_{\varepsilon,m,2} := 2(m-1)(1+\varepsilon)/\varepsilon, \\ \beta_{\varepsilon,m,3} = -\beta_{\varepsilon,m,4} = (m-1)/\varepsilon, \\ p_{\varepsilon,m}^{\{i\}}(t) := (-1)^{i-1} q_m(\alpha_{\varepsilon,m,i} t + \beta_{\varepsilon,m,i}), \qquad i = 1,2,3,4, \\ p_\varepsilon^{\{5\}}(t) := \chi_{[1/2,1/2+\varepsilon)}(t) - \chi_{[1/2-\varepsilon,1/2)}(t) \end{gathered}$$

and

$$\delta^{\{\varepsilon,m\}}(t) := \sum_{i=1}^{4} p_{\varepsilon,m}^{\{i\}}(t) + p_\varepsilon^{\{5\}}(t)$$

Applying Theorem 3.2 the following can be obtained:

Theorem 3.3. [21] *Let $m \geq 2$ be an integer, and let $\varepsilon > 0$ be such that*

$$B_\varepsilon := \{4[\frac{\sqrt{2}\,\varepsilon}{\sqrt{2}-1}\{(\varepsilon+1)(2m-2) + \frac{3}{\sqrt{2}}\}]^{\frac{1}{2}} + [\frac{\sqrt{2}\,\varepsilon}{\sqrt{2}-1}\{8\varepsilon + 4 + \frac{3}{\sqrt{2}}\}]^{\frac{1}{2}}\}^2 < 1.$$

Let $\psi_H^{\{\varepsilon,m\}} := \psi_H + \delta^{\{\varepsilon,m\}}$. Then

(a) $\psi_H^{\{\varepsilon,m\}} \in C^{m-2}(-\infty,\infty)$.

(b) *supp* $\psi_H^{\{\varepsilon,m\}} \subseteq [-\varepsilon, 1+\varepsilon]$.

(c) $\psi_H^{\{\varepsilon,m\}}$ *is a Riesz wavelet with Riesz bounds* $(1-\sqrt{B_\varepsilon})^2$ *and* $(1+\sqrt{B_\varepsilon})^2$.

(d) *If* $0 < \delta < \infty$, *then*

$$||\psi_H^{\{\varepsilon,m\}} - \psi_H||_{L^\delta(\mathbb{R})} \leq (1+2^\delta)^{1/\delta}(2\varepsilon)^{1/\delta}.$$

The functions $\psi_H^{\{\varepsilon,m\}}$ are called *perturbed Haar wavelets*. Since they are not orthonormal wavelets, the fast wavelet transform cannot be used to compute the wavelet coefficients. However, in [44] an algorithm is given for the fast computation of the wavelet coefficients, provided that $\varepsilon = 2^{-k}, k \in \mathbb{Z}^+$. The underlying justification for the method is that, as explained above, the perturbations are crafted by excising intervals around the discontinuities of the Haar functions and replacing them with functions based on B–splines of designated smoothness. The resulting functions can be calculated in $O(n)$ steps by relying on multilevel relations among the B–splines and interval integrals on the sections of the Haar functions which have not been excised. Due to the lack of orthogonality, there is still no completely satisfactory solution to the problem of reconstructing the original function. Since Riesz bounds are given explicitly, the frame algorithm or other algorithms can be used to reconstruct a function from its wavelet coefficients ([17, 22, 23]). The approach used by Schuler for the sine–Gabor frame wavelet that he introduced in [39] is to simply treat the wavelet as if it were orthogonal. Since his frames are nearly tight, this seems to work fine. But of course, one has no control over the error.

The remaining question about the perturbed Riesz wavelets is whether they are associated with an MRA. This was the motivation for some of the work cited in the next section.

4. Characterizations of orthonormal and Riesz wavelets.

Recall that MRA wavelets were defined at the beginning of Section 3.

If we know that a function is an orthonormal wavelet, how can we tell whether it is associated with an MRA? This question is answered using the *dimension function* $D_\psi(x)$, defined by

$$D_\psi(x) := \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^j(x+k))|^2.$$

Let $\mathbf{F}_\psi$ denote the closure in $\ell^2(\mathbb{Z})$ of the linear span of $\{\Psi_j(x), j\in\mathbb{Z}\}$, where $\Psi_j(x) := \{\widehat{\psi}(2^j(x+k)); k\in\mathbb{Z}\}\in\ell^2(\mathbb{Z})$.

Theorem 4.1. [28, p. 363] *Suppose ψ is an orthonormal wavelet. Then the following assertions are equivalent:*

(a) *ψ is an MRA wavelet.*

(b) $D_\psi(x) > 0$ *a.e.*

(c) $D_\psi(x) = 1$ *a.e.*

(d) $\dim \mathbf{F}_\psi(x) = 1$ *a.e.*

Motivated by Theorem 4.1 and by the question of whether the perturbed Haar wavelets are associated with an MRA, the second author [45] began the study of conditions that determine whether a Riesz wavelet ψ is associated with an MRA.

We will say that ψ is *obtained* by a multiresolution analysis, if there is some multiresolution analysis $M = \{U_j; j\in\mathbb{Z}\}$ such that $\psi\in U_1$. In this case we also say that ψ is obtained by M. A set A is *essentially contained* in a set B, if $A\setminus B$ has Lebesgue measure 0, and that A is *essentially equal* to B, if both $A\setminus B$ and $B\setminus A$ have Lebesgue measure 0. The main result of [45] is the following:

Theorem 4.2. *Let ψ be a Riesz wavelet. Then the following propositions are equivalent:*

(a) *ψ is obtained by an MRA and supp$\{\widehat{\psi}(2x)\}$ is essentially equal to supp$\{\widehat{\psi}(x)\}$.*

(b) ψ *is obtained by an MRA and supp* $\{\widehat{\psi}(2x)\}$ *is essentially contained in supp* $\{\widehat{\psi}(x)\}$.

(c) *There is a* 2*–periodic function* g *such that* $g(x) \neq 0$ *and*

$$\widehat{\psi}(2x) = g(x)\widehat{\psi}(x) \qquad a.e. \tag{4.1}$$

(d)

$$\sum_{k\in\mathbb{Z}} |\widehat{\psi}(x+2k)|^2 > 0 \qquad a.e.,$$

and there is a 2*–periodic function* g *such that (4.1) is satisfied a.e.*

A multivariate version of Theorem 4.2 was obtained by Bownik [8].

We can now answer the question of whether the perturbed Haar wavelets $\psi_H^{\{\varepsilon,m\}}$ are obtained by an MRA. A nontrivial application of Theorem 4.2 yields:

Theorem 4.3. [45, Theorem 4.3] *Let* B_ε *be defined as in Theorem 3.3, let* $m \geq 2$ *be an integer, and let* $\varepsilon > 0$ *be such that* $B(\varepsilon, m) < 1$. *Then* $\psi_H^{\{\varepsilon,m\}}$ *cannot be obtained by an MRA.*

The question now is what relationship there is between MRA wavelets and wavelets obtained by an MRA. The definitions trivially imply that if ψ is an MRA Riesz wavelet, then it is a Riesz wavelet obtained by the same MRA. The question then, is whether the converse is true. This has been studied by Kim et al. [32] and completely solved by Bownik [8], who studied the problem in the multidimensional setting and showed that a Riesz wavelet obtained by an MRA is an MRA Riesz wavelet. The univariate version of his result is the following:

Theorem 4.4. [8, Corollary 3.8] *Suppose* ψ *is a Riesz wavelet obtained by an MRA. Then* ψ *is a biorthogonal wavelet associated with the same MRA.*

But since an MRA Riesz wavelet is obtained by the same MRA, it follows that every MRA Riesz wavelet is biorthogonal. However, this result is already known: it is a consequence of results of Kim et al. [30] in one dimension, and Larson et al. [33] in several dimensions.

In summation: a Riesz wavelet is associated with an MRA if and only if it is obtained by an MRA. Moreover, every MRA Riesz wavelet is biorthogonal.

Basically, Theorem 4.2 gives necessary and sufficient conditions for a Riesz wavelet to be an MRA Riesz wavelet if $\widehat{\psi}(x) \neq 0$ a. e. For arbitrary Riesz wavelets another condition was given in [45, Proposition 2.2]. Kim et al. [32, Proposition 1.1] noted that, since a Riesz wavelet is an MRA Riesz wavelet if and only if it is associated with an MRA, this result can be expressed as follows:

Proposition 4.1. *A Riesz wavelet* ψ *is associated with an MRA if and only if there exist* $p, q \in L^2(-1/2, 1/2)$ *and* 1*–periodic, and* $\varphi \in L^2(\mathbb{R})$ *such that:*

(a) $\{\varphi_k; k \in \mathbb{Z}\}$ *is a Riesz sequence.*

(b) $\widehat{\varphi}(2x) = p(x)\widehat{\varphi}(x) \qquad a.e.$

(c) $\widehat{\psi}(2x) = q(x)\widehat{\varphi}(x) \qquad a.e.$

We say that a closed subspace S of $L^2(\mathbb{R})$ is *shift invariant*, if $\varphi \in S$ implies that $\varphi_k \in S$ for every $k \in \mathbb{Z}$. The theory of invariant subspaces is discussed in [25].

If $\{V_j; j \in \mathbb{Z}\}$ is an MRA, W_r will denote the orthogonal complement of V_r in V_{r+1}. Thus, $V_{r+1} = V_r \oplus W_r$.

Kim and his collaborators have also shown that Theorem 4.2 is a corollary of the following generalization of Theorem 4.1:

Theorem 4.5. [30, Theorem 2.6] *Suppose ψ is a Riesz wavelet. For $k \in \mathbb{Z}$, let $W_k := \overline{span}\{\psi_{k,l}; l \in \mathbb{Z}\}$. Then the following assertions are equivalent:*

(a) *ψ is an MRA Riesz wavelet.*

(b) $\sum_{k<0} W_k$ *is shift–invariant and* $\dim \mathbf{F}_\psi(x) = 1 \quad$ *a.e.*

(c) $\sum_{k<0} W_k$ *is shift–invariant and* $\dim \mathbf{F}(x) \geq 1 \quad$ *a.e.*

(d) $\sum_{k<0} W_k$ *is shift–invariant and* $D_\psi(x) > 0 \quad$ *a.e.*

The definitions of Riesz wavelets *associated* with an MRA and of Riesz wavelets *obtained* by an MRA can be carried over *verbatim* to frame wavelets. However in [36], Paluszyński et al. have introduced a different definition of MRA frame wavelet involving so–called *pseudo–scaling functions*, whereas in [37] they introduced the concept of MSA tight frame wavelets, also involving pseudo–scaling functions. Since frame theory is beyond the scope of this article, suffice it to say that the exact relationship between these different types of wavelet frames is still unclear.

The next result, due to Auscher, shows that an orthonormal wavelet that is continuous and satisfies certain decay conditions, is associated with an MRA having a continuous scaling function that also satisfies some decay conditions:

Theorem 4.6. *Let h be an orthonormal wavelet. Assume that $\widehat{h}$ is continuous on $\mathbb{R}$ and that there are constants $\alpha, C > 0$, such that*

$$|\widehat{h}(x)| \leq C|x|^\alpha (1 + |x|)^{-1/2-\alpha} \tag{4.2}$$

Then h is associated with an MRA. Moreover, this MRA has a scaling function φ such that $\widehat{\varphi}$ is continuous everywhere on $\mathbb{R}$, and there is a constant $D > 0$ such that

$$|\widehat{\varphi}(x)| \leq D(1 + |x|)^{-1/2-\alpha}. \tag{4.3}$$

A proof of this theorem is sketched in [2]. It is a particular case of [3, Theorem 1.3], which is a much more general multivariate result. Whether the results of [3] can be generalized to Riesz bases still seems to be an open question.

So far in this section, we have studied the following problem: If we know that a function ψ is a Riesz wavelet, how do we determine whether it is associated with an MRA? We will now focus on a slightly different issue: given an arbitrary function in $L^2(\mathbb{R})$, how do we determine whether it is an MRA Riesz wavelet? In other words, we need to determine whether the function satisfies two properties: that it is a Riesz wavelet *and* that it is associated with an MRA.

Gripenberg [24], Hernández, Wang and Weiss [27], and Wang [41] (see also [42]) have characterized the set of all orthonormal wavelets associated with a given MRA.

Theorem 4.7. [28, p. 57]. *If φ is an orthonormal scaling function for an MRA $\{V_j; j \in \mathbb{Z}\}$ and p is the associated low pass filter, then h is an orthonormal wavelet in W_0 if and only if there is a measurable unimodular and 1–periodic function $\nu(x)$, such that*

$$\widehat{h}(2x) = e^{2\pi xi}\,\nu(2x)\,\overline{p(x+1/2)}\,\widehat{\varphi}(x) \qquad a.e.$$

To determine whether a given function is an orthonormal wavelet associated with an MRA, we can use

Theorem 4.8. *A function $\psi \in L^2(\mathbb{R})$ is an MRA orthonormal wavelet if and only if the following four conditions are satisfied:*

(a) $\|\psi\| = 1$.

(b) $\displaystyle\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j x)|^2 = 1, \qquad a.e.$

(c) $\displaystyle\sum_{j\in\mathbb{Z}^+} \widehat{\psi}(2^j x)\,\overline{\widehat{\psi}(2^j(x+m))} = 0, \qquad a.e., \qquad m \in 2\mathbb{Z}+1.$

(d) $D_\psi(x) = 1 \qquad a.e.$

Here $D_\psi(x)$ is the dimension function defined at the beginning of this section.

Note that conditions (a), (b) and (c) are the same as in Theorem 1.3; they guarantee that ψ is an orthonormal wavelet. Adding condition (d) guarantees that ψ is associated with an MRA (cf. [28, p. 355, Theorem 3.2]).

Recent work by Hernández, Labate and Weiss [26] greatly extends Theorem 4.8, not only to a multivariate setting, but also to a more general class of functions that encompasses affine and Gabor wavelets as particular cases.

5. Perturbation in the Frequency Domain

Since the perturbed Haar wavelets are not associated with an MRA, the question remains as to whether it is possible to find an MRA Riesz wavelet having the same properties as the perturbed Haar wavelets. Further insight has been provided by Aimar et al., who have pointed out that Theorem 3.3 is a consequence of the following

Theorem 5.1. [1] *Let m be a nonnegative integer and let φ be such that its m^{th} derivative belongs to $L(\mathbb{R})$ and $\int_{\mathbb{R}} \varphi(t)\,dt = 1$. Let $\varepsilon > 0$ be such that*

$$M_\varepsilon := 2\sqrt{3}\,\varepsilon(1 + ||\varphi||_1)^2(4\sqrt{2}+1)(\sqrt{2}-1)^{-1}(4\varepsilon+1) < 1,$$

and let $\varphi_\varepsilon(t) := (1/\varepsilon)\varphi(t/\varepsilon)$. Then $\psi_H^\varepsilon := \psi_H * \varphi_\varepsilon$ belongs to $C^m(\mathbb{R})$, has support in $[-\varepsilon, 1+\varepsilon]$, and is a Riesz wavelet with bounds $(1-\sqrt{M_\varepsilon})^2$ and $(1+\sqrt{M_\varepsilon})^2$.

From Theorem 5.1 we see that adding a Bessel wavelet to the Haar wavelet ψ_H to obtain a Riesz wavelet is equivalent to multiplying $\widehat{\psi_H}$ by a suitable function μ. This suggests the idea of trying to multiply the Fourier transform of an orthonormal wavelet by a suitable function, so as to obtain a Riesz wavelet associated with a *different* MRA.

We begin with two auxiliary propositions. the first one is a direct consequence of Theorem 2.2:

Proposition 5.1. [14, Proposition 7.3.6 (i)]. *Let $\varphi \in L^2(\mathbb{R})$, and assume that $\{\varphi_k; k \in \mathbb{Z}\}$ is a frame sequence but not a Riesz sequence (i.e., it is overcomplete). Then $\Phi(x)$ is discontinuous.*

The following assertion is proved by an argument similar to the one used in the proof of [14, Proposition 7.3.6 (ii)].

Proposition 5.2. *Assume that $\varphi \in L^2(\mathbb{R})$, that $\widehat{\varphi}$ is continuous on $\mathbb{R}$, and that there are constants $D > 0$ and $\alpha > 0$ for which (4.3) is satisfied. If $\{\varphi_k; k \in \mathbb{Z}\}$ is a Riesz sequence, then $\Phi(x)$ is continuous in $\mathbb{R}$.*

Proof. Assume that (4.3) is satisfied, and let $x \in I$. Then, for all $n \in \mathbb{Z}^+$,

$$\begin{aligned}\left|\Phi(x) - \sum_{|k|\le n} |\widehat{\varphi}(x+k)|^2\right| &= \sum_{|k|>n} |\widehat{\varphi}(x+k)|^2 \\ &\le D^2 \sum_{|k|>n} (1+|x+k|)^{-1-2\alpha} \\ &\le 2D^2 \sum_{k>n} k^{-1-2\alpha}.\end{aligned}$$

This implies that the series

$$\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(x+k)|^2$$

is uniformly convergent. Thus $\Phi(x)$ is the uniform limit of continuous functions, and the assertion follows. □

Motivated by a paper of Aldroubi and Unser [4] we are now going to develop the idea of convolving Riesz bases, scaling functions, and multiresolution analyses. The first result involves convolving two suitably behaved functions, only one of which generates a Riesz basis of translates.

Theorem 5.2. *Assume that $\varphi \in L^2(\mathbb{R})$, that $\widehat{\varphi}$ is continuous on $\mathbb{R}$, and that there are numbers $D > 0$ and $\alpha > 0$ for which (4.3) is satisfied. Let $u \in L^2(\mathbb{R})$ be such that $\widehat{u}$ is continuous and bounded on $\mathbb{R}$, and $\widehat{u}(x) \neq 0$ whenever $\widehat{\varphi}(x) \neq 0$. Assume, moreover, that $\mathbf{S} := \{\varphi_k; k \in \mathbb{Z}\}$ is a Riesz sequence. If $\sigma := u * \varphi$, then $\mathbf{T} := \{\sigma_k; k \in \mathbb{Z}\}$ is a Riesz sequence.*

Theorem 5.2 is a corollary of the following generalization of [14, Proposition 7.3.9].

Theorem 5.3. *Assume that $\varphi \in L^2(\mathbb{R})$, that $\widehat{\varphi}$ is continuous on $\mathbb{R}$, and that there are numbers $D > 0$ and $\alpha > 0$ for which (4.3) is satisfied. Let v be a continuous and bounded function in $\mathbb{R}$ such that $v(x) \neq 0$ whenever $\widehat{\varphi}(x) \neq 0$. Assume, moreover, that $\mathbf{S} := \{\varphi_k; k \in \mathbb{Z}\}$ is a Riesz sequence. If $\widehat{\sigma}(x) := v(x)\widehat{\varphi}(x)$, then $\mathbf{T} := \{\sigma_k; k \in \mathbb{Z}\}$ is a Riesz sequence.*

Proof. Let A and B be a lower and an upper Riesz bound for S respectively. From Proposition 5.2 we know that $\Phi(x)$ is continuous. Applying Theorem 2.2 we therefore conclude that

$$A \le \Phi(x) \le B \qquad \text{for every } x \in I. \tag{5.1}$$

Thus, for every $x \in I$

$$\Phi_1(x) := \sum_{k\in\mathbb{Z}} |\widehat{\sigma}(x+k)|^2 = \sum_{k\in\mathbb{Z}} |v(x+k)\widehat{\varphi}(x+k)|^2 \tag{5.2}$$
$$\leq ||v||_\infty^2 \Phi(x) \leq ||v||_\infty^2 B.$$

To find a lower bound for $\Phi_1(x)$ we proceed as follows: For $m \in \mathbb{Z}$ and $\delta > 0$, let

$$U(m,\delta) := \{x : -1 < x < 2 \text{ and } |\widehat{\varphi}(x+m)|^2 > \delta\},$$

and let $\mathfrak{F}$ denote the family of all these sets $U(m,\delta)$. From (5.1) we deduce that for every $x \in I$ there are numbers $m \in \mathbb{Z}$ and $\delta > 0$ such that $x \in U(m,\delta)$. Thus $\mathfrak{F}$ is a an open covering of I, and Heine–Borel's lemma implies that there is a finite subcollection of $\mathfrak{F}$, say $\{U(m_n,\delta_n); 1 \leq n \leq N\}$, that also covers $\mathfrak{F}$. Let F_n denote the closure of $U(m_n,\delta_n)$. Since $F_n \subset [-1,2]$, we see that $\{F_n; 1 \leq n \leq N\}$ is a family of compact sets that covers I.

If $x \in F_n$, then $|\widehat{\varphi}(x+m_n)|^2 \geq \delta_n$. Since F_n is compact, the hypotheses imply there is a number $\varepsilon_n > 0$ such that $|v(x+m_n)|^2 \geq \varepsilon_n$. Clearly

$$\delta := \min\{\delta_n; 1 \leq n \leq N\} > 0 \qquad \text{and} \qquad \varepsilon := \min\{\varepsilon_n; 1 \leq n \leq N\} > 0.$$

Let $x \in I$. Then there is an integer n, $1 \leq n \leq N$, such that $x \in F_n$. Thus

$$\Phi_1(x) \geq |\widehat{\sigma}(x+m_n)|^2 = |v(x+m_n)\widehat{\varphi}(x+m_n)|^2 \geq \delta\varepsilon. \tag{5.3}$$

Combining (5.2), (5.3), and Theorem 2.2, the assertion follows. □

The convolution of scaling functions was studied as early as 1987 by Micchelli and Prautzsch [35]. The most general result is due to Wang:

Theorem 5.4. [41, Theorem 2.47] *Let φ_1 and φ_2 be orthonormal scaling functions (not necessarily for the same MRA), with low pass filters p_1 and p_2 respectively. Let $\varphi(x) := \varphi_1 * \varphi_2$, and let $r(x)$ be defined by (2.1). Assume that*

$$\sum_{k\in\mathbb{Z}} |\widehat{\varphi_1}(x+k)\widehat{\varphi_2}(x+k)| > 0, \qquad a.e., \tag{5.4}$$

and let

$$\widehat{\sigma}(x) := \frac{\widehat{\varphi_1}(x)\widehat{\varphi_2}(x)}{r(x)}. \tag{5.5}$$

Then σ is an orthonormal scaling function with low pass filter

$$p(x) := p_1(x)p_2(x)\frac{r(x)}{r(2x)}. \tag{5.6}$$

Condition (5.4) may be difficult to verify directly, since it involves an infinite sum. Here is a theorem that may be easier to apply.

Theorem 5.5. *Let φ_1 and φ_2 be orthonormal scaling functions with low pass filters p_1 and p_2 respectively. Assume, moreover that both $\widehat{\varphi_1}$ and $\widehat{\varphi_2}$ are continuous on $\mathbb{R}$, that $\widehat{\varphi_2}$ satisfies an inequality of the form (4.3) for some $D > 0$ and $\alpha > 0$, and that $\widehat{\varphi_1}(x) \neq 0$ whenever $\widehat{\varphi_2}(x) \neq 0$. Let σ and p be defined by (5.5) and (5.6) respectively. Then σ is an orthonormal scaling function with low pass filter p.*

Proof. Proceeding as in the proof of Theorem 5.2 we see that $\Phi(x)$ satisfies an inequality of the form (5.3), i.e, there is a $C > 0$ such that $\Phi(x) > C$, and the assertion follows from Theorem 5.4. □

We now turn to the convolution of multiresolution analyses. We begin with a consequence of Theorem 5.4.

Theorem 5.6. *Let h_1 and h_2 be MRA orthonormal wavelets with orthonormal scaling functions φ_1 and φ_2, and low pass filters p_1 and p_2 respectively. Let $\varphi(x) := \varphi_1 * \varphi_2$, assume that (5.4) is satisfied, let $r(x)$ be given by (2.1), and*

$$\mu(x) := e^{-\pi x i}\frac{r(x/2+1/2)}{r(x/2)r(x)}, \tag{5.7}$$

and let σ and p be defined by (5.5) and (5.6) respectively. If

$$\widehat{h}(x) := \widehat{h_1}(x)\widehat{h_2}(x)\mu(x), \tag{5.8}$$

then h is an MRA orthonormal wavelet with orthonormal scaling function σ, and low pass filter p.

Proof. Applying Theorem 4.7 we see that there are unimodular 1–periodic functions $\nu_1(x)$ and $\nu_2(x)$ such that

$$\widehat{h_1}(2x) = e^{2\pi x i}\,\nu_1(2x)\,\overline{p_1(x+1/2)}\,\widehat{\varphi_1}(x) \qquad \text{a.e.}$$

and

$$\widehat{h_2}(2x) = e^{2\pi x i}\,\nu_2(2x)\,\overline{p_2(x+1/2)}\,\widehat{\varphi_2}(x) \qquad \text{a.e.}$$

Since $r(x)$ is 1–periodic, a straightforward computation yields

$$\widehat{h_1}(2x)\widehat{h_2}(2x) = e^{2\pi x i}\,\nu(2x)\,\overline{p(x+1/2)}\,\widehat{\varphi}(x)\frac{e^{2\pi x i}r(2x)}{r(x+1/2)} \qquad \text{a.e.},$$

where $\nu(x) = \nu_1(x)\nu_2(x)$. Thus

$$\widehat{h}(2x) = e^{2\pi x i}\,\nu(2x)\,\overline{p(x+1/2)}\,\widehat{\sigma}(x) \qquad \text{a.e..}$$

From Theorem 5.4 we know that σ is an orthonormal scaling function with low pass filter p, and the assertion follows by another application of Theorem 4.7. $\square$

Since the hypotheses of Theorem 5.5 imply that (5.4) is satisfied, Theorem 5.6 yields

Theorem 5.7. *Let h_1 and h_2 be MRA orthonormal wavelets with orthonormal scaling functions φ_1 and φ_2, and low pass filters p_1 and p_2 respectively. Assume, moreover that both $\widehat{\varphi_1}$ and $\widehat{\varphi_2}$ are continuous on $\mathbb{R}$, that $\widehat{\varphi_2}$ satisfies an inequality of the form (4.3) for some $D > 0$ and $\alpha > 0$, and that $\widehat{\varphi_1}(x) \neq 0$ whenever $\widehat{\varphi_2}(x) \neq 0$. Let σ, p and h be defined by (5.5), (5.6) and (5.8) respectively. Then h is an MRA orthonormal wavelet with orthonormal scaling function σ and low pass filter p.*

To determine whether h is an MRA orthonormal wavelet using Theorem 5.7, we need certain information on the scaling functions φ_1 and φ_2. The following theorem bypasses this requirement.

Theorem 5.8. *Let h_1 and h_2 be orthonormal wavelets. Assume that $\widehat{h_1}$ and $\widehat{h_2}$ are continuous on $\mathbb{R}$, and that there are constants $\alpha_1, \alpha_2, C_1, C_2 > 0$ such that*

$$|\widehat{h_i}(x)| \leq C_i|x|^{\alpha_i}(1+|x|)^{-1/2-\alpha_i}, \qquad i = 1, 2.$$

Assume, moreover that

$$\sum_{k\in\mathbb{Z}^+} |\widehat{h_1}(2^j x)|^2 \neq 0 \quad \textit{whenever} \quad \sum_{k\in\mathbb{Z}^+} |\widehat{h_2}(2^j x)|^2 \neq 0.$$

Then h_1 and h_2 are MRA wavelets. If φ_1 and φ_2 are their scaling functions and h, σ and p are defined by (5.5), (5.6) and (5.8), respectively, then h is an MRA orthonormal wavelet with orthonormal scaling function σ, and low pass filter p.

Proof. By Theorem 4.6 h_1 and h_2 are MRA wavelets, $\widehat{\varphi_1}$ and $\widehat{\varphi_2}$ have continuous Fourier transforms, and there are constants $D_1, D_2 > 0$, such that

$$|\widehat{\varphi}_i(x)| \le D_i(1+|x|)^{-1/2-\alpha_i}, \qquad i = 1, 2.$$

We want to establish a connection between the zeros of the scaling functions φ_i and the zeros of the wavelets h_i, so we can apply Theorem 5.5. We first need to establish a representation for the scaling functions in terms of the wavelets.

From([28, p. 61]) we know that

$$|\widehat{\varphi}_i(x)|^2 = \sum_{j\in\mathbb{Z}^+} |\widehat{h}_i(2^j x)|^2 \qquad \text{a.e} \tag{5.9}$$

Define

$$u_i(x) := \left(\sum_{j\in\mathbb{Z}^+} |\widehat{h}_i(2^j x)|^2\right)^{1/2}, \qquad i = 1, 2.$$

From the hypotheses we know that

$$|\widehat{h}_i(2^j x)| \le C_i|2^j x|^{\alpha_i}(1+|2^j x|)^{-1/2-\alpha_i}, \qquad i = 1, 2.$$

Thus, if $|x| \ge \delta > 0$ we see that

$$|\widehat{h}_i(2^j x)| \le C_i\delta^{-1/2}2^{-j/2}, \qquad i = 1, 2,$$

and proceeding as in the proof of Proposition 5.2, we readily see that the functions $u_i(x)$ are continuous on $|x| \ge \delta$. Since δ is arbitrary, this implies that the functions $u_i(x)$ are continuous on $\mathbb{R} \setminus \{0\}$. Since the functions $\widehat{\varphi}_i$ are continuous on $\mathbb{R}$, (5.9) implies that

$$|\widehat{\varphi}_i(x)| = u_i(x), \qquad x \in \mathbb{R}\setminus\{0\}, \qquad i = 1, 2.$$

Since Theorem 1.2 implies that

$$\widehat{\varphi}_i(0) \neq 0, \qquad i = 1, 2,$$

we conclude that $\widehat{\varphi_1}(x) \neq 0$ whenever $\widehat{\varphi_2}(x) \neq 0$, and the assertion follows from Theorem 5.7. □

Example. The following example illustrates the application of Theorem 5.5, Theorem 5.6, Theorem 5.7, and Theorem 5.8.

Let φ_1 denote the characteristic function of the interval $[0, 1)$. The following facts are well known [17, 28]: The Haar function ψ_H is an MRA orthonormal wavelet, and there is an MRA associated with ψ_H that has φ_1 as its scaling function. Moreover,

$$\widehat{\varphi_1}(x) = e^{-\pi x i}\frac{\sin \pi x}{\pi x}$$

Let $\widehat{h}_1(t) := \widehat{h}_2(t) := \psi_H(t)$. Since

$$\widehat{\psi_H}(x) = ie^{-\pi x i}\frac{\sin^2(\pi x/2)}{\pi x/2}, \tag{5.10}$$

we see that the hypotheses of Theorem 5.8 are satisfied. Thus, if

$$\widehat{h}(x) := \widehat{h_1}(x)\widehat{h_2}(x)\mu(x), \tag{5.11}$$

where $\mu(x)$ is given by (5.7), we know that h is an MRA orthonormal wavelet.

If we require more information, we may apply Theorem 5.6 or Theorem 5.7 instead:

If $\varphi := \varphi_1 * \varphi_1$, then

$$\widehat{\varphi}(x) := \widehat{\varphi_1}(x)\widehat{\varphi_1}(x) = \widehat{N_2}(x) = e^{-2\pi xi}\left(\frac{\sin \pi x}{\pi x}\right)^2.$$

Since

$$\Phi(x) = \sum_{k\in\mathbb{Z}} |\widehat{N_2}(x+k)|^2,$$

from, e. g., [15, (4.2.10)], we conclude that

$$[r(x)]^2 = \Phi(x) = 1/3 + (2/3)\cos^2 \pi x > 0.$$

This implies that the conditions of Theorem 5.6 are satisfied, and therefore that the function h defined by (5.11) is an MRA orthonormal wavelet.

Let us now find an explicit representation for $\widehat{h}$.

By direct computation,

$$\mu(x) = \sqrt{3}e^{-\pi xi}\left[\frac{1+2\sin^2 \pi x/2}{(1+2\cos^2 \pi x/2)(1+2\cos^2 \pi x)}\right]^{1/2}.$$

Thus, applying (5.10) we have:

$$\widehat{h}(x) = [\widehat{\psi_H}(x)]^2\mu(x) =$$

$$-\sqrt{3}e^{-3\pi xi}\frac{\sin^4(\pi x/2)}{(\pi x/2)^2}\left[\frac{1+2\sin^2 \pi x/2}{(1+2\cos^2 \pi x/2)(1+2\cos^2 \pi x)}\right]^{1/2}.$$

We can also verify our conclusions by a third method: The function

$$\widehat{\psi}(x) := \sqrt{3}e^{-\pi xi}\frac{\sin^4(\pi x/2)}{(\pi x/2)^2}\left[\frac{1+2\sin^2 \pi x/2}{(1+2\cos^2 \pi x/2)(1+2\cos^2 \pi x)}\right]^{1/2}$$

is a Battle–Lemarié wavelet, which is known to be an MRA orthonormal wavelet (cf. [17, p. 148]). Since $\widehat{h}(x)$ is the product of $\psi(x)$ and a unimodular 1–periodic function, applying Theorem 4.7 we deduce that h is an MRA orthonormal wavelet.

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