Annuities, Insurance and Life

by

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These notes cover a standard one-term course in Actuarial Mathematics with a primer from the Theory of Interest. The notes are to be presented at Auburn University, Fall 2003, as part of the Actuarial Mathematics sequence. Furthermore, this is a work in progress, hopefully converging over the course of the term. Students, please be patient (this is saving you big bucks), and please offer questions and comments concerning the material. This will help me formulate a presentation that will presumably be more accessible to others.

Thanks, Pat Goeters

Chapter 1

An Introduction to Interest and Mortality

1.1 Life and Death

The premise that must be remembered in this course is that money is effected by interest rates. The basic subject of this course can be summarized in two sentences. We will consider the value now of a payment made in the future by investigating the question; "how much is it worth to us now, to be given \$1 t years from now?" We then add the stipulation that the payment is made contingent upon a certain event occurring, and then examine the question.

A fundamental issue in the insurance business is to be able to estimate when an accident will occur, a stock crashes, or when an insured passes. It is common practice in the study of insurance from an actuarial stand-point to lump these things together. We are interested in the probability (the percentage of time) that a *life aged* x expires or dies within t years. Often this is softened by referring to a life aged x as *decrementing* within t years. From the insured's side, we wish to know the probability that a life aged x *lives* or *survives* at least t years.

If a company has been selling a policy, they will periodically review the rates. In this case, they have some experience about lives aged x (say female drivers 21 years old, driving a Nissan Pathfinder). If our company's experience is ample, we can estimate with some degree of certainty the portion of lives aged x that will decrement (get into an accident). If our company's experience is inadequate or the policy has not been offered before, as actuaries, we may seek data from the actuaries at a company selling a similar product. A third option is to develop an educated guess as to a mathematical model (formula) for the probabilities - a few standard models will be given later.

The careful student has noted that our data set is finite but as is the often the case, we will assume that *mortality* is continuous. That is, we assume that a life age x may expire at any positive time t (not just at year's end say), and yet the chance of anyone expiring at the exact time t is zero. Insurance policies are not payable at the moment of decrement though, so the companies often will not be concerned with the exact time of decrement only the chance of decrement within a given year (or month).

The *curtate-future-life* of a life age x is the whole number of years completed before death. Using this, the chance of a life age 30 having curtate-future-life 3, say, may not be zero. In summary, we will assume that we have a continuous model of mortality though often we will limit consideration to discrete (yearly, monthly) times.

The probability that a life aged x survives at least t years is denoted by ${}_tp_x$. In building our data table we start with a given population, and watch its development over time. The only way for a member to leave the population is through decrement and, no new members can be added to the population. We refer to members of the population as *lives*, and ℓ_t is the number of lives aged t. Then, the probability that a life aged x lives t or more additional years is just

$${}_tp_x = \frac{\ell_{x+t}}{\ell_x},$$

provided we have the numbers ℓ_x, ℓ_{x+t} at our fingers.

An important special case of the symbolism ${}_tp_x$ and ${}_tq_x$ is the case when $t = 1$. In this case, the 1 is not written, and

$$p_x = \text{the probability } (x) \text{ survives at least one year},$$

and

$$q_x = \text{the probability } (x) \text{ dies before the first year is up}.$$

We represent the generic or average life aged x by (x). The probability that (x) decrements before t years is denoted by ${}_tq_x$ and by elementary reasoning

$${}_tp_x +_t q_x = 1.$$

Also, of the ℓ_{x+t} lives aged $x+t$, q_{x+t} represents the percentage of them that die with the year. I.e.,

$$d_{x+t} = \ell_{x+t}\, q_{x+t}$$

is the number of lives aged $x+t$ that die before reaching age $x+t+1$. So, ${}_tp_x\, q_{x+t} = \frac{d_x}{\ell_x}$ is the probability that a life aged x lives to be $x+t$ but dies before age $x+t+1$.

The Illustrative Life Table I is a widely known and referenced mortality table. It was derived using certain *assumptions* on mortality using census data for the U.S.

1.2 Introduction to Interest

In the case of money placed into a savings account at your local bank, your money is affected positively by the interest the bank affords and negatively by the inflation rate. Of course, if $\imath$ is the given interest rate per period, then \$1 now will be worth $1+\imath$ dollars one period hence. A point of departure is whether or not the interest computation for periods $2, 3, \ldots$ involves the interest earned in previous periods.

Simple interest is said to apply if the interest computation is only applied to the original sum of money invested (note that borrowing money can be viewed as an investment by the lender). For example, under simple interest, an amount P, accumulates to $P + P\imath = P(1+\imath)$ at the end of one period, and accumulates to $P\imath + P(1+\imath) = P(1+2\imath)$ at the end of two periods, and so on. At the end of n periods, P accumulates to $P(1+n\imath)$.

In this course, we will eschew simple interest in favor of the common practice of using *compound interest.* For compound interest, the interest computation at the end of a given period involves the interest already accrued or earned in previous periods as part of the balance. Under compound interest, P dollars today is worth

$$P(1+\imath)^n = P(1+\imath)^{n-1} + P(1+\imath)^{n-1}\imath$$

n periods hence. The factor $1+\imath$ is called the *accumulation factor* in that multiplying a balance P by $(1+\imath)^n$ allows us to accumulate P forward in time n years (with cumulative interest).

When the interest conversion period is a year and $\imath$ is the interest rate for the year, $\imath$ is called the *effective rate* of interest. The effective rate is an annual rate of interest. Sometimes it is convenient to adopt an interest conversion period other than a year (a month say) as is the case with most conventional loans, and we may wish refer to the *effective rate of interest for a given conversion period*. However, most often in the text and the literature, we simply regard the standard conversion period to be 1 year, under the realization that no loss of generality occurs since we can easily replace the year with another conversion period if warranted.

Exercises :

1.1 Using the ILT I, calculate ${}_{25}p_{40}$, ${}_{20}q_{20}$, ${}_{40}p_{20}$, ${}_{50}q_{0}$.

1.2 Using the ILT I, calculate the probabilities that

(a) (20) dies when they are 30.

(b) a 1 year old lives to age 21.

(c) a 12 year old dies in their 20's.

(d) a person age 47 dies within a year.

1.3 If the interest rate is $\imath = .06$ per year, how long will it take for an amount of money to double under interest compounding with rate $\imath$? Give an equation which answers this question for general $\imath$.

Chapter 2

The Theory of Interest

2.1 Introduction to Interest

We will assume that we are given an effective rate $\imath$ which is an annual rate. A *nominal interest* rate $\jmath$ refers to an annual interest rate where interest calculation or conversion is done more frequently than once a year (but is still compounded). It has been my experience that whenever you are quoted an interest rate, you assume that it is a nominal rate, converted monthly. For example, the interest rate usually quoted for an automobile loan or other conventional loans with monthly payments (while credit cards compute interest on a daily basis). We use the phrase *interest is compounded* to replace "interest is converted under compounding of interest".

When the annual rate $\jmath$ is given as a nominal rate compounded m^{th}ly, what is meant is that interest is compounded at the rate $\frac{\jmath}{m}$, every m^{th} of the year. (The effective rate for each m^{th} is then $\frac{\jmath}{m}$.) So, at the end of k number of m^{th}'s of a year, P accumulates to $P(1+\frac{\jmath}{m})^k$ by the formula for compound interest.

We say that a nominal rate $\jmath$ is *equivalent* to the effective rate $\imath$ if over any number of years, 1 dollar accumulates to the same amount under each plan for interest computation. Assume that $\jmath$ is a nominal rate with m^{th}ly compounding. If $\jmath$ is equivalent to an effective rate $\imath$, then

$$1+\imath = (1+\frac{\jmath}{m})^m.$$

Solving for $\jmath$, we have

$$\jmath = m[(1+\imath)^{1/m} - 1].$$

Conversely, if $\jmath = m[(1+\imath)^{1/m} - 1]$, then $\jmath$ satisfies $1 + \imath = (1 + \frac{\jmath}{m})^m$ and \$1 accumulates to

$$(1+\imath)^t = (1 + \frac{\jmath}{m})^{mt},$$

after t years under each form of interest conversion. So, $m[(1+\imath)^{1/m} - 1]$ compounded m^{th}ly is equivalent to $\imath$.

The symbol $\imath^{(m)}$ represents the nominal interest rate $m[(1+\imath)^{1/m} - 1]$ (with compounding every m^{th} of the year) that is equivalent to $\imath$. This results in the relational formulas

$$\imath = (1 + \frac{\imath^{(m)}}{m})^m - 1 \quad \text{and} \quad \imath^{(m)} = m[\sqrt[m]{1+\imath} - 1].$$

We've seen that if you start with P dollars and apply the effective rate $\imath$, the balance at the end of n years is $P(1+\imath)^n$. Conversely, if you have Q dollars n years after a deposit of P dollars, then $P(1+\imath)^n = Q$, so that $P = \frac{1}{(1+\imath)^n} Q = \nu^n Q$, where $\nu = \frac{1}{1+\imath}$. We call $\nu = \frac{1}{1+\imath}$ the *discount factor* corresponding to $\imath$. To reiterate, in order to have P dollars n years from now, you must invest $\nu^n P$ today. Just as the accumulation factor $1+\imath$ allows us to move money forward, the discount factor $\nu = \frac{1}{1+\imath}$ allows us to draw money back in time; Q dollars today was worth $\nu^n Q$ at the time exactly n years ago.

The number $\delta = \lim_{m\to\infty} \imath^{(m)}$ has meaning as the *interest rate assuming continuous compounding* corresponding to the effective rate $\imath$. We call δ the *force of interest*. By L'Hospital's Rule, $\imath^{(m)} = \frac{(1+\imath)^{1/m} - 1}{1/m}$ converges to

$$\delta = ln(1+\imath) = -ln\ \nu.$$

It follows that the accumulation factor is then

$$1 + \imath = e^{\delta},$$

and therefore the balance of a loan (or the accumulation of savings), t years after the initial loan (or deposit) of amount P - assuming no adjustments have been made - is

$$Pe^{\delta t} = P(1+\imath)^t,$$

which we interpret as correct even when t is not a whole number.

Conversely, the discount factor is

$$\nu = e^{-\delta},$$

so an amount P, t years forward from now is worth $Pe^{-\delta t}$ at present.

Observe, we now have that $\imath^{(m)} = m[e^{\delta/m} - 1]$. If we define a function $f(x) = x[e^{\delta/x} - 1]$, then

$$f'(x) = e^{\delta/x}(1 - \frac{\delta}{x}) - 1.$$

Reasonable bounds on $\imath$ insure that $\delta < 1$. The Maclaurin series for $e^{-(\delta/x)}$ is

$$e^{-(\delta/x)} = 1 - \frac{\delta}{x} + \frac{1}{2}\frac{\delta^2}{x^2} - \frac{1}{3!}\frac{\delta^3}{x^3} + \cdots .$$

Since $\frac{\delta}{x} < 1$, the Maclaurin series above has absolutely monotonically decreasing, alternating terms and so $e^{-(\delta/x)} > 1 - \frac{\delta}{x}$. Hence, $1 > e^{(\delta/x)}(1 - \frac{\delta}{x})$ and consequently $f'(x) < 0$. That is, the $\imath^{(m)}$'s are monotonically decreasing as m increases.

We have established

$$\dagger \quad \imath = \imath^{(1)} > \imath^{(2)} > \imath^{(3)} > \ldots > \delta > 0.$$

Although most loans and savings plans are formulated using an effective or nominal rate plan, an analogous practice is not uncommon in the issuing of bonds, that of paying the interest up front. When borrowing P dollars, the bank may offer take a percentage d at the beginning of an interest cycle (in lieu of collecting interest at the end), so that at the end of 1 year, the remaining debt is P; d is called the *effective rate of discount* and is an annual rate.

For example, on August 7^{th}, 2003, a U.S. EE Patriot Bond sold for \$25 and matures to \$50; that is, the government borrows \$50 from you less a 50% discount, in order to pay you back \$50 at the maturity date.

The effective rate $\imath$ associated with d can be found in terms of d as follows (in the EE Patriot Bond example, $\imath = 2.66\%$): Using the accumulation factor $1 + \imath$,

$$(1 + \imath)(1 - d)P = P,$$

from which we obtain

$$(1 + \imath)(1 - d) = 1 \quad \text{and} \quad \imath = \frac{1}{1 - d} - 1 \quad \text{and} \quad d = 1 - \nu = \frac{\imath}{1 + \imath}.$$

So, d is the ratio of the interest earned in 1 year to the balance after 1 year (as defined in some books).

The *nominal rate of discount* $d^{(m)}$ is an annual rate such that a discount rate of $\frac{d^{(m)}}{m}$ is applied at the beginning of each m^{th} of the year. It follows that if $\imath$ is the effective rate for $d^{(m)}$, then

$$(1+\frac{\imath^{(m)}}{m})(1-\frac{d^{(m)}}{m})=1,$$

and so, for example,

$$d^{(m)}=m[1-\nu^{1/m}].$$

In a manner analogous to obtaining †, we obtain

$$d=d^{(1)}<d^{(2)}<d^{(3)}<\ldots<\delta,$$

and

$$\delta=\lim_{m\to\infty}d^{(m)}.$$

2.2 Annuities-Certain

A regular *annuity* or *annuity-certain* is any timed series of payments in which the payments are certain to be paid. Examples include payments on your financed automobile, mobile home, or house, as well as the paying off of credit card balances. The term "certain" refers to the legal obligation rather than a force of will. If you cease making payments, the lender will recoup the outstanding balance in some other way and/or write it off as a loss, marking the debt "paid". There is another type of annuity, called a life-annuity, that will be considered in Chapter 3, but at this point there will be no confusion if we refer to an annuity-certain as simply an annuity.

While, conceivably, the payments can occur at various times we will stick to consideration of the standard annuities where the payments are equal and are made at equal increments. In order to evaluate these payments, as we've learned, we must pick a set point in time in which to calculate payments. The term "present time" refers to the arranged time in which the payment schedule goes into effect (which is not necessarily the time the payments start).

We will now cover the standard annuities. Most of the standard annuities are *level*, meaning that the payments are equal, and in all, the scheduled time between payments is the same. We refer to an annuity as *discrete* is the payments are made either annually or at some other fixed points in time.

Annuity-Immediate

In this situation it is assumed that a series of payments is made in unit amount (\$1) at the end of every period (or year) and that $\imath$ is the effective rate for that period (or year). The "immediate" modifier conveys that the payments are to be made at the end of the period (or year), perhaps contrary to a direct reading of the word immediate. Also, it is common to assume that payments cease at some point, say n periods (or years).

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The *present value of a payment* is the value of the payment when discounted back to the present. For example, the present value to the first payment is ν. The present value of the second payment is ν^2, and so on. Reasonably, the *present value of an annuity* is the sum of the present values of all of the payments. The present value of an annuity-immediate payable for n years is therefore $\nu + \nu^2 + \cdots + \nu^n$.

The symbol $a_{\overline{n}|}$ represents the present value of an annuity-immediate payable in unit amount for n years in which the frequency of payments coincides with the interest conversion period. Thus,

$$a_{\overline{n}|} = \nu + \nu^2 + \cdots + \nu^n = \nu(1 + \nu + \cdots + \nu^{n-1}) = \nu(\frac{1-\nu^n}{1-\nu}) = \frac{1-\nu^n}{\imath}.$$

Again,

$$a_{\overline{n}|} = \nu + \nu^2 + \cdots + \nu^n = \frac{1-\nu^n}{\imath}.$$

Note: If payments are P instead of \$1 in our annuity-immediate, then the present value of the annuity, i.e., the present value of all payments, is

$Pa_{\overline{n}|}$. What this means is that the certainty of payments in amount P made according to the schedule described is worth $Pa_{\overline{n}|}$ presently.

For example, suppose you wish to purchase a car and you learn that after the down payment (if any), taxes and fees the cost will be \$18,731. You are told that you can be financed at the annual rate of 5.99% for 5 years. Having lived in this world for some time, you realize you are to make monthly payments in some amount P to pay off your vehicle. What may not be clear until you read the fine print is that the quoted 5.99% is a nominal rate, convertible (or, to be compounded) monthly. So the effective rate $\imath$ for each month is $\imath = \frac{.0599}{12}$ and the number of payments is $n = 60$. The present value of payments must equal the loan amount. So,

$$\$18,731 = Pa_{\overline{60}|} \quad \text{and} \quad P = \$362.04.$$

This payment is about a half-penny over the true amount $\frac{\$18,731}{a}_{\overline{60}|}$ (please

excuse the "=" above) so the final payment is usually adjusted to make things even.

Annuity-Due

In this situation it is assumed that a series of payments is made in unit amount (\$1) at the beginning of every period (or year) and that $\imath$ is the effective rate for that period (or year). The word "due" conveys that payments are made at the beginning of the year. Again, it is often assumed that payments cease after some fixed number of years, and we will let n be the number of years (=payments).

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The present value of an annuity-due payable for n years in unit amount is denoted by $\ddot{a}_{\overline{n}|}$, and as above we find that

$$\ddot{a}_{\overline{n}|} = 1 + \nu + \cdots + \nu^{n-1} = \frac{1-\nu^n}{1-\nu} = \frac{1-\nu^n}{d}.$$

As before, if the payments are P instead of \$1 in our annuity-immediate, then the present value of the annuity is $P\ddot{a}_{\overline{n}|}$.

For example, suppose you wish to lease a car for 3 years and that after the capital reduction (if any), taxes and fees the cost will be \$18, 731. You are quoted an interest rate of 5.99% for your lease which you interpret to be a nominal rate convertible monthly. If you are told that the payments are $P = \$355.00$ per month, payable in advance (as is the convention with leases). What is the residual value of the car in 3 years? That is, what will be the purchase price left to you in case you wish to buy the vehicle when the lease runs out?

The present value of your payments is $P\ddot{a}_{\overline{36}|} = \$11,729.20$ (to the nearest penny). This leaves \$7, 001.80 unpaid in present dollars. Of course, your lease terminates 3 years forward from the present, so at the time of the lease-expiration, the residual payment will be $\$7,001.80(1+\frac{.0599}{12})^{36} = \$8,376.42$ to the nearest penny. This assumes that no funny business is being waged.

Annuities Payable m^{th}ly

Another set of standard annuities arises when \$1 is paid per annum but in installments of $\frac{1}{m}$ every m^{th} of the year. The payments are to last a fixed number of years, say n. Given the effective annual rate $\imath$, we calculate the nominal rate $\imath^{(m)}$.

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As in the case of annual payments, the payments may occur at the beginning or end of the payment period. An *annuity-immediate payable* m^{th}ly is such an annuity where payments are made at the end of each m^{th} of a year (again there is often a finite number of years for payments to be made, and we will consider this case and denote this number by n). Using the formula for a geometric sum again we can obtain a closed form for the present value of all payments payable for n years:

$$a_{\overline{n}|}^{(m)} = \frac{1}{m}\nu^{1/m} + \frac{1}{m}\nu^{2/m} + \frac{1}{m}\nu^{3/m} + \cdots + \frac{1}{m}\nu^{nm/m} = \frac{1}{m}\nu^{1/m}(\frac{1 - \nu^{nm/m}}{1 - \nu^{1/m}}),$$

and therefore, after a little algebra,

$$a_{\overline{n}|}^{(m)} = \frac{1 - \nu^n}{m((1+\imath)^{1/m} - 1)} = \frac{1 - \nu^n}{\imath^{(m)}}.$$

In words, the present value of an annuity-immediate payable m^{th}ly in annual unit amount for n years is $a_{\overline{n}|}^{(m)} = \frac{1-\nu^n}{\imath^{(m)}}$. If the payments are P per *year* instead of \$1, then the present value of payments is $Pa_{\overline{n}|}^{(m)} = P\frac{1-\nu^n}{\imath^{(m)}}$.

The standard *annuity-due payable* m^{th}*ly* is an annuity with payments of $\frac{1}{m}$ per m^{th} of the year, with payments occurring at the beginning of each m^{th}. Assuming the payments are to be made for n years, the present value of payments is denoted by $\ddot{a}_{\overline{n}|}^{(m)}$ and as above can written in closed form:

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1 - \nu^n}{d^{(m)}}.$$

Note: If P is the *annual* payment made under an annuity-due payable m^{th}ly for n years (so the m^{th}ly payments are $\frac{P}{m}$), then the present value is $P\ddot{a}_{\overline{n}|}^{(m)}$.

The annuity-immediate payable monthly is the most frequently encountered annuity for us. The car purchase and lease examples are rightfully annuities payable monthly. We can illustrate how a car or home purchase uses the concepts of an annuity-immediate payable monthly with another example:

Suppose you (including your parents perhaps) wish to buy a house off campus with the amount to be financed of $\$123,789$, on a 30 mortgage at an effective rate of 6.125%. This leads to an equivalent nominal rate of $\imath^{(12)} = 5.595\%$ and a discount factor of $\nu = .942285$. If P be the total of all payments made in a year (without adjusting for interest), then the present value of all payments must match the loan amount; i.e.,

$$P a_{\overline{30}|}^{(12)} = P \frac{1-\nu^{30}}{\imath^{(12)}} = \$123,789.$$

Therefore, $P = \$\frac{123{,}789}{13.982159} = \$8,853.36$ (they always round up), and so your monthly payments to pay off the house are $\frac{\$8{,}853.36}{12}$ or \$737.78. By the way, the amount \$737.78 is called your P+I payment. There are two other amounts that are often required to be payed to your mortgage company: if your loan-to-value ratio is more than 80%, then you must pay PMI (mortgage insurance) which may be around \$100 per month; also, they usually want you to escrow taxes and home insurance, which in Auburn will run you another \$100 per month. While PMI is money lost to you, escrowing taxes and insurance are just easy ways for you to pay those bills.

Although we've found out whether or not we can swing the mortgage there are other aspects of the loan that prove interesting. Suppose we want to sell the house in t years. How much will we still owe the mortgage company? This is a good time to go over the *Prospective and Retrospective Methods* of evaluation.

Sitting at time t (in years), immediately after any payment for year t, we may look forward (there-by adopting a prospective view):

$$|\text{—}|\text{—}|\text{—}\cdots|\text{—}|\text{—}|\text{—}\cdots\text{—}|$$

We know that we have some outstanding balance on the loan, call the outstanding balance B_t. Most loans are configured according to the *Equivalence Principle*; that is, the assumption that our future payments pay off the remaining balance. We assume we have such a loan. (I have never seen a loan that wasn't based on the Equivalence Principle, but I have seen several loans emphasize that they are based on the Equivalence Principle). Hence, the present value of all payments made subsequent to year t, $Pa^{(12)}_{\overline{n-t|}}$, must equal B_t:

$$B_t = (B_0/a^{(12)}_{\overline{n|}})a^{(12)}_{\overline{n-t|}} = B_0\frac{1-\nu^{n-t}}{1-\nu^n}.$$

For example, when $t = 5$, $\frac{1-\nu^{n-t}}{1-\nu^n}$ is roughly 93%, and so (regardless of what house you are buying) you have paid off 7% of your loan. Couple that with a real estate growth rate in Auburn sometimes in the 5% − 10% per year range, you are likely to see a cash equity of over 30% of your original purchase price after 5 years.

In contrast to a prospective approach, at time t, we can look *back* to evaluate our progress on the loan. This is called a *retrospective method of evaluation*. We might address the question, "what is our outstanding balance after t years?", immediately following the last of the payments for year t, retrospectively. To do this we need some standard terminology.

The symbol a in the annuity symbols $a_{\overline{n|}}$, $\ddot{a}_{\overline{n|}}$, $a^{(m)}_{\overline{n|}}$, and $\ddot{a}^{(m)}_{\overline{n|}}$ refers to the present value of payments. Each of the standard annuity symbols refers to a n-year term annuity. Instead of valuing the payments at present, we can also value the payments at a future time; specifically, at the end of n years, after the last payment has been made.

For the n-year term annuity-immediate,

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the sum of the payments viewed n years past the present is

$$1 + (1 + \imath) + \cdots (1 + \imath)^{n-1} = \frac{(1+\imath)^n - 1}{1+\imath-1} = \frac{(1+\imath)^n - 1}{\imath}.$$

This sum is called the *accumulated value of payments* of an annuity-immediate and is denoted by $s_{\overline{n|}}$. Note also, from general reasoning, that

$$s_{\overline{n|}} = (1 + \imath)^n a_{\overline{n|}}.$$

Similarly,

$$\ddot{s}_{\overline{n|}} = \frac{(1+\imath)^n - 1}{d} \quad \text{and} \quad \ddot{s}_{\overline{n|}} = (1+\imath)^n \ddot{a}_{\overline{n|}},$$

and

$$s_{\overline{n}|}^{(m)} = \frac{(1+\imath)^n - 1}{\imath^{(m)}} \qquad \text{and} \qquad s_{\overline{n}|}^{(m)} = (1+\imath)^n a_{\overline{n}|}^{(m)},$$

and

$$\ddot{s}_{\overline{n}|}^{(m)} = \frac{(1+\imath)^n - 1}{d^{(m)}} \qquad \text{and} \qquad \ddot{s}_{\overline{n}|}^{(m)} = (1+\imath)^n \ddot{a}_{\overline{n}|}^{(m)}.$$

Back to a retrospective evaluation of a loan with scheduled payments at the end of each m^{th}. We start with a loan amount of $B_0 = Pa_{\overline{n}|}^{(m)}$ (in general), and make mt payments in t years, of P/m at the end of each m^{th}. At the end of year t, after the payments for the t^{th} year have been completed, the accumulated value of all payments to year t is

$$Ps_{\overline{t}|}^{(m)},$$

and the accumulated original loan value is

$$(1+\imath)^t Pa_{\overline{n}|}^{(m)}.$$

Therefore, we still owe

$$B_t = (1+\imath)^t Pa_{\overline{n}|}^{(m)} - Ps_{\overline{t}|}^{(m)} = P\frac{(1+\imath)^t - \nu^{n-t} - ((1+\imath)^t - 1)}{\imath^{(m)}},$$

at time t, which can be rewritten as,

$$B_t = P\frac{1-\nu^{n-t}}{\imath^{(m)}} = Pa_{\overline{n-t}|}^{(m)},$$

as obtained by the prospective evaluation.

Deferred Annuities, Perpetuities, Continuous and Increasing/Decreasing Annuities

A *perpetuity* is an annuity for which payments never cease. In the conventional notation the symbol ∞ replaces the symbol n. Then

$$a_{\overline{\infty}|} = \lim_{n\to\infty} a_{\overline{n}|} = \frac{1}{i},$$

$$\ddot{a}_{\overline{\infty}|} = \lim_{n\to\infty} \ddot{a}_{\overline{n}|} = \frac{1}{d},$$

$$a_{\overline{\infty}|}^{(m)} = \lim_{n\to\infty} a_{\overline{n}|}^{(m)} = \frac{1}{\imath^{(m)}},$$

and

$$\ddot{a}_{\overline{\infty}|}^{(m)} = \lim_{n\to\infty} \ddot{a}_{\overline{n}|}^{(m)} = \frac{1}{d^{(m)}}.$$

Just as we may consider interest to be compounded continuously, we are able to consider an n-year term annuity, *payable continuously* in unit annual amount. Interpreting this concept, if $a_{\overline{n}|}$ is the present values of annuity payments, where the *bar* conveys continuity, then

$$a_{\overline{n}|} \equiv \lim_{m\to\infty} \Sigma_{k=1}^{nm} \frac{1}{m} e^{-\frac{k}{m}\delta} = \int_0^n e^{-\delta t} dt.$$

Alternately, one could define,

$$a_{\overline{n}|} \equiv \lim_{m\to\infty} a_{\overline{n}|}^{(m)} = \lim_{m\to\infty} \frac{1-\nu^n}{\imath^{(m)}} = \frac{1-\nu^n}{\delta} = \int_0^n e^{-\delta t} dt.$$

Of course this is the same as $\lim_{m\to\infty} \ddot{a}_{\overline{n}|}^{(m)}$. The perpetuity payable continuously in unit annual amount is

$$\overline{a}_{\overline{\infty}|} = \int_0^\infty e^{-\delta t} dt = \frac{1}{\delta}.$$

A *deferred annuity* refers to an annuity where the payments are postponed or deferred until some future time. The fact that an annuity has been deferred m years is conveyed by having the symbol $m|$ appear in the lower left of an annuity symbol. For example, the n-year term, k-year deferred annuity-due in annual unit amount has present value

$${}_{k|}\ddot{a}_{\overline{n}|},$$

while the present value of a k-year deferred perpetuity payable in m^{th}'s in annual unit amount is written as

$${}_{k|}a_{\overline{\infty}|}^{(m)}.$$

Do you see that the present value of a k-year deferred annuity is just ν^k times the present value of the annuity where the payments begin at present? E.g.,

$${}_{k|}\ddot{a}_{\overline{n}|} = \nu^k \ddot{a}_{\overline{n}|}, \quad {}_{k|}a_{\overline{\infty}|} = \nu^k a_{\overline{\infty}|}. \quad {}_{k|}a_{\overline{n}|}^{(m)} = \nu^k a_{\overline{n}|}^{(m)},$$

for just a few examples.

As you've no doubt recognized, each of the standard annuities we have examined so far are *level* in that the payments are all the same. There are two important variations on the level annuities and those are the *increasing*, and the *decreasing* annuities.

The increasing annuities vary according to whether the payments occur at the beginning or end of the year or whether payments are m^{th}ly. At this point in the course one should be able to decipher the meanings of the present values of annuities $(Ia)_{\overline{n}|}$, $(I\ddot{a})_{\overline{n}|}$, $(Ia)^{(m)}_{\overline{n}|}$, $(\overline{Ia})_{\overline{n}|}$, $(I\ddot{a})^{(m)}_{\overline{n}|}$, $(\overline{Ia})_{\overline{n}|}$ and $(\overline{Ia})_{\overline{\infty}|}$ once one has been introduced to a single one.

The symbol $(Ia)^{(m)}_{\overline{n}|}$ represents the present value of an annuity-immediate payable m^{th}ly for n years, for which the annual premium in the t^{th} year is \$$t$. So,

$$(Ia)^{(m)}_{\overline{n}|} = \frac{1}{m}\nu^{1/m} + \frac{1}{m}\nu^{2/m} + \cdots + \frac{1}{m}\nu^{m/m} + \frac{2}{m}\nu^{(m+1)/m} + \frac{2}{m}\nu^{(m+2)/m} +$$

$$\cdots + \frac{2}{m}\nu^{2m/m} + \cdots + \frac{n}{m}\nu^{(m(n-1)+1)/m} + \frac{n}{m}\nu^{(m(n-1)+2)/m} + \cdots + \frac{n}{m}\nu^{nm/m}.$$

There are various manipulations that can be made of this expression, however none lead to simple expressions. For example, we can read the above equation as

$$(Ia)^{(m)}_{\overline{n}|} = a^{(m)}_{\overline{n}|} + {}_{1|}a^{(m)}_{\overline{n-1}|} + {}_{2|}a^{(m)}_{\overline{n-2}|} + \cdots + + {}_{n-1|}a^{(m)}_{\overline{1}|}.$$

The present value of the n-year increasing annuity payable continuously is

$$(\overline{Ia})_{\overline{n}|} = \int_0^n te^{-\delta t}dt,$$

can easily be computed using parts;

$$(\overline{Ia})_{\overline{n}|} = \frac{te^{-\delta t}}{-\delta}|_0^\infty + \frac{1}{\delta}\int_0^\infty e^{-\delta t}dt = \frac{1}{\delta^2}.$$

Analogously, one can interpret the symbolism $(Da)_{\overline{n}|}$, $(D\ddot{a})_{\overline{n}|}$, $(Da)^{(m)}_{\overline{n}|}$, $(D\ddot{a})^{(m)}_{\overline{n}|}$, and $(\overline{Da})_{\overline{n}|}$ once one understands that in the corresponding annuities, the annual payments for year t total $n-t$. One can choose to relate increasing and decreasing annuities (for example, $(D\ddot{a})_{\overline{n}|} = (n+1)\ddot{a}_{\overline{n}|} - (I\ddot{a})_{\overline{n}|}$), however we will not make much use of increasing and decreasing annuities in this course.

Another glimpse at the significance of the symbolism occurs when we understand that $(\overline{I}\,\overline{a})_{\overline{n}|}$ has a different meaning than $(I\overline{a})_{\overline{n}|}$. The existence of the bar over the I in the former term means that the annual premium in the annuity is continually increasing while the lack of the bar over the I in the latter expression means that the annual premium is increasing discretely. So,

$$(\overline{I}\,\overline{a})_{\overline{n}|} = \int_0^n te^{-\delta t}dt,$$

while

$$(I\overline{a})_{\overline{n}|} = \overline{a}_{\overline{n}|} + {}_{1|}\overline{a}_{\overline{n-1}|} + \cdots + {}_{n-1|}\overline{a}_{\overline{1}|} < (\overline{I}\,\overline{a})_{\overline{n}|}.$$

Exercises :

2.1 Given $\imath = .06$, compute the equivalent rates and factors d, ν, δ, $\imath^{(12)}$, and $d^{(12)}$.

2.2 Show that $\delta = \lim_{m\to\infty} d^{(m)}$.

2.3 Show that $d < d^{(2)} < d^{(3)} < \cdots$.

2.4 Suppose you are shopping for a new car. You want to make sure you can handle the monthly payments so before looking, you make a small table, using the nominal interest rates $\jmath = .039, .049, .059$ (convertible monthly), of the payment-cost per month per dollar financed assuming a 60 month loan. Make this table now (please).

2.5 Derive the expression $\ddot{a}^{(m)}_{\overline{n}|} = \frac{1-\nu^n}{d^{(m)}}$.

2.6 If you purchased a car for \$18,789.00 under financing with a 5.99% nominal rate for 60 months, how much do you need to sell the car for at the end of 3 years to get out from under the vehicle (i.e., pay off the loan exactly)?

2.7 Show that $\frac{\imath}{\delta} a_{\overline{n}|} = a_{\overline{n}|}$ and $d a_{\overline{n}|} = \imath \ddot{a}_{\overline{n}|}$.

2.8 If the effective annual rate $\imath$ is 7%, calculate $\imath^{(m)}$, δ, d, and $d^{(m)}$.

2.9 Suppose that a company knows that expenditures are forthcoming and wishes to plan for the expenses by investing money at present. If the expenses are \$25,000 next year for advertising, \$5,000 in two years for additional computers, and \$10,000 in five years for additional office space, how much must be invested today, at the effective rate 5.25% in order to meet these expenses (assume expenditures are made at the end of the year)?

2.10 An investor accumulates a fund by making payments at the beginning of each month for 6 years. Her monthly payment is 50 for the first 2 years, 100 for the next 2 years, and 150 for the last 2 years. At the end of the 7^{th} year the fund is worth 10,000. The effective annual interest rate is $\imath$ and the effective monthly rate is signified by j. Which of the following formulas represents the equation of value for this fund accumulation?

(a) $\ddot{s}_{\overline{24}|\imath}(1+\imath)[(1+\imath)^4+2(1+\imath)^2+3]=200.$
(b) $\ddot{s}_{\overline{24}|\imath}(1+j)[(1+\jmath)^4+2(1+\jmath)^2+3]=200.$
(c) $\ddot{s}_{\overline{24}|\jmath}(1+\imath)[(1+\imath)^4+2(1+\imath)^2+3]=200.$
(d) $s_{\overline{24}|\imath}(1+\imath)[(1+\imath)^4+2(1+\imath)^2+3]=200.$
(e) $\ddot{s}_{\overline{24}|\imath}(1+\jmath)[(1+\jmath)^4+2(1+\jmath)^2+3]=200.$

2.11 Matthew makes a series of payments at the beginning of each year for 20 years. The first payment is 100. Each subsequent payment through the tenth year increases by 5% from the previous payment. After the tenth payment, each payment decreases by 5% from the previous payment. Calculate the present value of these payments at the time the first payment is made using an effective annual rate of 7%.

(a) 1375
(b) 1385
(c) 1395
(d) 1405
(e) 1415

2.12 Megan purchases a perpetuity-immediate for 3250 with annual payments of 130. At the same price and interest rate, Chris purchases an annuity-immediate with 20 annual payments that begin at amount P and increase by 15 each year thereafter. Calculate P.

(a) 90

(b) 116
(c) 131
(d) 176
(e) 239

2.13 For 10,000, Kelly purchases an annuity-immediate that pays 400 quarterly for the next 10 years. Calculate the annual nominal interest rate convertible monthly earned by Kelly's investment.

(a) 10.0%
(b) 10.3%
(c) 10.5%
(d) 10.7%
(e) 11.0%

2.14 Payments of X are made at the beginning of each year for twenty years. These payments earn interest at the end of the year at an annual effective rate of 8%. The interest is immediately reinvested at an annual effective rate of 6%. At the end of twenty years, the accumulated value of the twenty payments and the reinvested interest is 5600. Calculate X.

(a) 121.67
(b) 123.56
(c) 125.72
(d) 127.18
(e) 128.50

2.15 The present value of a 25-year annuity-immediate with a first payment of 2500 and decreasing by 100 each year thereafter is X. Assuming an annual effective interest rate of 10%, calculate X.

(a) 11,346
(b) 13,615

(c) 15,923

(d) 17,396

(e) 18,112

2.16 Given one of the rates, determine other.

(a) $d = .03$, $\imath^{(2)} =?$

(b) $\delta = .05$, $d =?$

(c) $\nu = .942671$, $d^{(2)} =?$

(d) $\imath = .06$, $\imath^{(12)} =?$

2.17 Check which formula is correct. If incorrect, give the correct expression.

(a) $d\imath\nu\imath d - \nu\imath\nu\imath d = d\imath d$

(b) $d^{(12)} = 12[e^{(-\delta/12)} - 1]$

(c) $\imath^{(2)} \cdot d^{(2)} = \imath^{(2)} - d^{(2)}$

(d) $\delta = ln\sqrt{\frac{1-d}{1+\imath}}$

2.18 Check which formula is correct. If incorrect, give the correct expression.

(a) $\ddot{a}^{(m)}_{\overline{n}|} = \frac{1}{m}[\frac{1-(1+\imath)^{-n}}{1-(1+\imath)^{1/m}}]$

(b) $(Is)_{\overline{n}|} - (I\ddot{s})_{\overline{n}|} = \ddot{s}_{\overline{n}|}$

(c) $a_{\overline{m+n}|} - a_{\overline{m}|} = \nu^{m-r}[\ddot{a}_{\overline{r+n}|} - \ddot{a}_{\overline{n}|}]$

(d) $\frac{a_{\overline{n-t}|}}{a_{\overline{n}|}} + \frac{s_{\overline{t}|}}{s_{\overline{n}|}} = 0$

The next three problems refer to a series of payments of 10, payable at the beginning of each quarter for 10 years. Correct (if necessary) the proposed value for the present value of payments using a minimal alteration.

2.19 Assuming $d^{(4)} = .04$, the present value is $\frac{10[1-(.99)^{40}]}{1-(.99)^{1/4}}$.

2.20 Assuming $\imath^{(3)} = .06$, the present value is $\frac{10[1-(1.02)^{-30}]}{1-(1.02)^{-1/3}}$.

2.21 Assuming $d^{(3)} = .06$, the present value is $\frac{10[1-(.98)^{30}]}{1-(.98)^{3/4}}$.

2.22 Which of the following expression does not represent a definition of $a_{\overline{n}|}$?

(a) $\nu^n[\frac{(1+\imath)^n-1}{\imath}]$

(b) $\frac{1-\nu^n}{\imath}$

(c) $\nu + \nu^2 + \cdots + \nu^n$

(d) $\nu[\frac{(1+\imath)^n-1}{\nu}]$

(e) $\frac{s_{\overline{n}|}}{(1+\imath)^n}$

2.23 An estate provides a perpetuity with payments of X at the end of each year. Susan, Seth, and Lori share the perpetuity such that Seth receives the payments of X for the first n years and Susan receives the payments of X for the next m years, after which Lori receives all the remaining payments of X.

Which of the following represents the difference between the present value of Seth's and Susan's payments using a constant rate of interest?

(a) $X[a_{\overline{n}|} - \nu^n a_{\overline{m}|}]$

(b) $X[\ddot{a}_{\overline{n}|} - \nu^n \ddot{a}_{\overline{m}|}]$

(c) $X[a_{\overline{n}|} - \nu^{n+1} a_{\overline{m}|}]$

(d) $X[a_{\overline{n}|} - \nu^{n+1} a_{\overline{m}|}]$

(e) $X[\nu a_{\overline{n}|} - \nu^{n+1} a_{\overline{m}|}]$

2.24 Mike receives cash flows of 100 today, 200 in one year, and 100 in two years. The present value of these cash flows is 364.46 at an annual effect rate of interest $\imath$.

Calculate $\imath$.

(a) 10%

(b) 11%

(c) 12%

(d) 13%

(e) 14%

2.25 A loan is being repaid with 25 annual payments of 300 each. With the tenth payment, the borrower pays an extra 1000, and the repays the balance over 10 years with a revised annual payment. The effective rate of interest is 8%.

Find the amount of the revised annual payment.

(a) 157
(b) 183
(c) 234
(d) 257
(e) 383

2.26 The present value of a series of 50 payments starting at 100 at the end of the first year and increasing by 1 each year thereafter is equal to X. The annual effective interest rate is 9%.

Calculate X.

(a) 1165
(b) 1180
(c) 1195
(d) 1210
(e) 1225

2.27 Which of the following are characteristics of all perpetuities?

I. The present value is equal to the first payment divided by the annual effective interest rate.
II. Payments continue forever.
III. Each payment is equal to the interest earned on the principal.

(a) I only
(b) II only
(c) III only
(d) I, II, and III

(e) The correct answer is not given by (a), (b), (c), or (d).

2.28 At a nominal interest rate i convertible semi-annually, an investment of 1000 immediately and 1500 at the end of the first year will accumulate to 2600 at the end of the second year.

Calculate i.

(a) 2.75%
(b) 2.77%
(c) 2.79%
(d) 2.81%
(e) 2.83%

2.29 An annuity immediate pays 20 per year for 10 years, then decreases by 1 per year for 19 years. At an annual effective interest rate of 6%, the present value is equal to X.

Calculate X.

(a) 200
(b) 205
(c) 210
(d) 215
(e) 220

2.30 At an annual effective rate of $\imath$, the present value of a perpetuity-immediate starting with a payment of 200 in the first year and increasing by 50 each year thereafter is $46,530$.

Calculate $\imath$.

(a) 3.25%
(b) 3.50%
(c) 3.75%
(d) 4.00%
(e) 4.25%

2.31 Calculate the nominal rate of discount convertible monthly that is equivalent to a nominal rate of interest of 18.9% per year convertible monthly.

(a) 18.0%
(b) 18.3%
(c) 18.6%
(d) 18.9%
(e) 19.2%

2.32 An investor wishes to accumulate $10,000$ at the end of 10 years by making level deposits at the beginning of each year. The deposits earn a 12% annual effective rate of interest paid at the end of each year. The interest is immediately reinvested at an annual effective interest rate of 8%.

Calculate the level deposit.

(a) 541
(b) 572
(c) 598
(d) 615
(e) 621

2.33 A discount electronics store advertises the following financial arrangement:

"We don't offer you confusing interest rates. We'll just divide your total cost by 10 and you can pay us that amount each month for a year."

The first payment is due on the date of the sale and the remaining 11 payments at monthly intervals thereafter.

Calculate the effective annual interest rate the store's customers are paying on their loans.

(a) 35.1%

(b) 41.3%
(c) 42.0%
(d) 51.2%
(e) 54.9%

2.34 An annuity pays 1 at the end of each year for n years. Using an effective annual interest rate of $\imath$, the accumulated value of the annuity at time $n+1$ is 13.776. It is also known that $(1+\imath)^n = 2.476$.

Calculate n.

(a) 4
(b) 5
(c) 6
(d) 7
(e) 8

2.35 A bank customer takes out a loan of 500 with a 16% nominal interest rate convertible quarterly. The customer makes payments at the end of each quarter.

Calculate the amount of principal in the fourth payment.

(a) 0.0
(b) 0.9
(c) 2.7
(d) 5.2
(e) There is not enough information to calculate the amount of principal.

In the next four problems, derive the given formula:

2.36 $(I\,a)_{\overline{n}|} = \frac{1}{\imath}(\ddot{a}_{\overline{n}|} - n\nu^n)$.

2.37 $(I\,\ddot{a})_{\overline{n}|} = \frac{1}{d}(\ddot{a}_{\overline{n}|} - n\nu^n)$.

2.38 $(I\,a)_{\overline{n}|} + (D\,a)_{\overline{n}|} = (n+1)\,a_{\overline{n}|}$.

2.39 $(I\ddot{a})_{\overline{n}|} + (D\ddot{a})_{\overline{n}|} = (n+1)\ddot{a}_{\overline{n}|}$.

2.40 Verify the following: In an n-year level annuity-immediate in unit amount payable annually, the present value of future payments at the end of the m^{th} year is $a_{\overline{n-m}|}$. The interest paid in the m^{th} years is $\imath \cdot a_{\overline{n-m+1}|}$ and the amount of the payment in the m^{th} year that goes towards the principal is $1 - \imath \cdot a_{\overline{n-m+1}|}$.

2.41 At present, investments are made at the beginning of each of 8 years, at an annual interest rate of 10%. If the initial investment is \$100 which is increases by \$50 each year thereafter, then

(a) how much is in the investment, immediately after the last payment?

(b) how much is in the investment, eight years past the present?

Chapter 3

The Probability of Life

3.1 Elementary Probability

We assume throughout that there is a mortality function ${}_tp_x$. In the second section we will attempt to estimate ${}_tp_x$ provided we know its values for t, k integers. Here is a sample of easily derived identities which may prove useful later:

$$ {}_{s+t}p_x ={}_t p_x \cdot_s p_{x+t} \quad , \quad {}_{s|t}p_x ={}_{s+t} p_x $$

and

$$ {}_{s|t}q_x = \ {}_sp_x \ {}_tq_{x+s} = \ {}_{t+s}q_x - \ {}_sq_x. $$

The last string of equalities gives the probability that a life aged x lives s years, then dies within the next t years. An important quantity in our subject is the probability that a life (x) dies in the $k+1^{st}$ year. This can be written as

$$ {}_kp_x \cdot q_{x+k} \quad \text{and} \quad {}_kp_x - \ {}_{k+1}p_x \quad \text{and also} \quad {}_{k+1}q_x - \ {}_kq_x. $$

The *curtate-expectation-of-life* of (x), denoted by e_x, is understood as a measure in some sense of how long (x) might "expect" to live in round years - though the answer is not necessarily in round years. In statistics, if 50% of a population lives 30 years and 50% lives 20 years, it is said that a member of the population "expects" to live 25 years in that the mathematical (weighted) average of years lived is 25. In general the formula for the curtate-expectation-of-life is:

$$e_x = \Sigma_{k=0}^{\infty} k \cdot_k p_x \; q_{x+k}.$$

We will now examine the probability function ${}_tp_x$ for fixed x and continuous t. Again, the chance of decrementing precisely at t is zero. However we can look at the chance of (x) decrementing in an interval $[t, t+\varepsilon]$; we get

$${}_tp_x \; {}_\varepsilon q_{x+t} =_t p_x({}_\varepsilon q_{x+t}/\varepsilon) \cdot \varepsilon = \frac{{}_{t+\varepsilon}q_x \; -_t q_x}{\varepsilon} \cdot \varepsilon.$$

Taking the limit as $\varepsilon \to 0$ gives us an expression which figuratively represents the notion of (x) living t years then dying suddenly.

Set

$$\mu(x+t) \equiv (\lim_{\varepsilon \to 0} \frac{{}_{t+\varepsilon}q_x \; -_t q_x}{\varepsilon})/{}_tp_x = (\frac{d}{dt} \; {}_tq_x)/{}_tp_x.$$

Then, the expression ${}_tp_x\mu(x+t)dt$ symbolically represents the notion that (x) lives t years, ${}_tp_x$, then dies before age $t+dt$. This justifies the moniker *force of mortality* on (x) at year t for $\mu(x+t)$.

By general reasoning, $\int_0^s {}_tp_x \; \mu(x+t)dt$ is the "sum" of the probabilities that (x) lives t years then dies suddenly, summing over all $0 \leq t \leq s$; i.e.,

$${}_sq_x = \int_0^s {}_tp_x\mu(x+t)dt,$$

This identity also follows from the Fundamental Theorem of Calculus since ${}_tp_x\mu(x+t) = \frac{d \; {}_tq_x}{dt}$. Thus,

$${}_sp_x = 1 -_s q_x = \int_0^\infty {}_tp_x\mu(x+t)dt - \int_0^s {}_tp_x\mu(x+t)dt = \int_s^\infty {}_tp_x\mu(x+t)dt.$$

By asserting

$$1 = \int_0^\infty {}_tp_x\mu(x+t)dt = \lim_{t \to \infty} {}_tq_x,$$

we are merely assuming that "everyone must die sometime" which we accept without further ado. In fact it is SOP to assume there is a *limiting age* denoted by ω beyond which nobody lives.

3.2 Fractional Age Assumptions and Analytical Laws

Given mortality data (for example, the data from the ILT, or, experience gathered from our Claims Department), it is common practice to to lump all lives with the same whole number of years lived, together. We do not write of a life aged 26.2, rather (26). This bolsters information on (x) when x is an integer, and from this information we can calculate ${}_kp_x = \frac{\ell_{x+k}}{\ell_x}$ for k, x integers. But we still need to have ${}_tp_x$ available when t is not an integer (since decrementing is not only an annual occurrence), and there are several widely accepted ways to interpolate between our data ${}_kp_x$ for k, x integers.

Recall that

$$ {}_kp_x = p_x \cdot_{k-1} p_{x+1} = p_x \cdot p_{x+1} \cdots p_{x+k-1}, $$

so the essential data is p_x where x is an integer. Also, if k is the greatest integer in t and $s = t - k$, then

$$ {}_tp_x =_k p_x \cdot_s p_{x+k}, $$

and so what we need to estimate is ${}_sp_x$ for $0 < s < 1$.

Uniform Distributions of Death

Under the *uniform distribution of deaths* or *UDD* for short, we perform linear interpolation between successive known death rates ${}_0q_x$ and q_x as an estimate for ${}_sq_x$ when $0 < s < 1$. That is, under UDD,

$$ {}_sq_x = s(q_x) + (1 - s)({}_0q_x) = sq_x \quad \text{for} \;\; 0 < s < 1. $$

(You have to have careful eyes.)

Constant Force of Mortality

Under the constant force assumption, it is assumed that for x a fixed integer, $\mu(x + s)$ is constant for all $0 < s < 1$. Specifically, $\mu(x + s) =$

$\mu(x)$ for all $0 < s < 1$. Then ${}_sp_x = e^{\int_0^s \mu(x+t)dt} = e^{\mu s}$ where $\mu = \mu(x)$.

Baldacci's Assumption

Named after the Italian actuary G. Baldacci, this assumption performs *hyperbolic interpolation* between ${}_xp_0$ and ${}_{x+1}p_0$, in that, we assume

$$\frac{1}{{}_{x+s}p_0} = \frac{(1-s)}{{}_xp_0} + \frac{s}{{}_{x+1}p_0}.$$

Since ${}_{x+t}p_0 ={}_x p_0 \cdot_t p_x$, this gives ${}_sp_x = \frac{{}_{x+s}p_0}{{}_xp_0}$ and so

$$\frac{1}{{}_sp_x} = \frac{{}_xp_0}{{}_{x+s}p_0} = \frac{(1-s){}_xp_0}{{}_xp_0} + \frac{s{}_xp_0}{{}_xp_0p_x} = (1-s) + \frac{s}{p_x}.$$

Thus, ${}_sq_x = 1 -_s p_x = \frac{sq_x}{1+(s-1)q_x}$ when all is said and done.

We will employ the UDD assumption in Chapter 5.

Another approach to fitting a continuous model to our experience or mortality data, is to choose a model that you have reason to believe describes population. An ASA from AFLAC once told me that he assumed cancer rates conformed to a well-known continuous model (I have forgotten which model). I asked him how he determined that (I was really curious). He said, "'cause the guy over at Prudential did it that way". Hey, we just try to do our best.

In all models, there is a limiting age ω. What is interesting with these well-known and widely used analytical laws of mortality is the date in which they were first used.

Some Analytical Laws of Mortality

De Moivre (1729)	${}_xp_0 = 1 - \frac{x}{\omega}$	$0 \leq x \leq \omega$
Gompert (1825)	${}_xp_0 = exp[-m(c^x - 1)]$	$0 \leq x, c > 1$
Makeham (1860)	${}_xp_0 = exp[-Ax - m(c^x - 1)]$	$0 \leq x, c > 1$

Weibull (1939)	${}_xp_0 = e^{-ux^{n+1}}$	$0 \leq x, n > 0$

Exercises

3.1 Use integration by parts to show that ${}_tp_x = exp[-\int_0^t \mu(x+s)ds]$.

3.2 The *complete-expected-life* of (x) is defined to be $\overset{\circ}{e}_x = \int_0^\infty t {}_tp_x \mu(x+t)dt$. Show that $\overset{\circ}{e}_x = \int_0^\infty {}_tp_x dt$.

3.3 Under UDD, show the following for integral x and $0 < s < 1$:

(a) $\mu(x+s) = \frac{q_x}{1-sq_x}$.

(b) ${}_sp_x\mu(x+s) = q_x$.

3.4 Under the constant force assumption, show the following for integral x and $0 < s < 1$:

(a) ${}_sp_x = p_x^s$.

(b) $\mu(x+s) = -ln\ p_x$.

Chapter 4

Annuities Payable Upon Survival

4.1 Life Annuities

A *life-annuity* is an annuity for which the payments are made contingent only upon survival (or lack of decrement). It would be bad form for Metropolitan Life (for example) to ask you to continue to make payments on your whole-life policy years after you are gone. The types of standard life-annuities mirror the regular annuities considered earlier but now we must factor in the probability that (x) survives to a given payment. The notation must convey this contingency.

Discrete Level Whole-Life Annuity Issued To (x):

The modifier *whole-life* indicates that payments are made as long as (x) survives. There are two discrete whole-life annuities issued to (x) payable annually; whole-life annuity-due, and whole-life annuity-immediate. While these are not perpetuities (everyone dies), there is no set term limiting the number of payments.

The *whole-life annuity-due* issued to (x) is an annuity payable, in level amounts, at the beginning of the year, as long as (x) is alive. A *whole-life annuity-immediate* issued to (x) is a level annuity, payable at the end of the year, contingent upon (x)'s survival.

In addition to using life-annuities to pay for life insurance (or as they say at AFLAC, life assurance), many retirement plans, like Alabama State Teacher's Retirement, or other government plans, are simply deferred life annuity-immediate payable monthly (with perhaps a partial payment at the end of the month of death which is determined by the percentage of the month retiree was alive).

The *whole-life annuity-due payable* m^{th}ly in annual unit amount, issued to (x) is an annuity payable in amount $\frac{1}{m}$ at the beginning of each m^{th} of the year, as long as (x) is alive. The unit *whole-life annuity-immediate payable* m^{th}ly issued to (x) is an annuity, payable in amount $\frac{1}{m}$ at the end of each m^{th} of the year, contingent upon (x)'s survival.

The *annuitant* is the person or entity making the payments prescribed by the annuity. In the Alabama State Teacher's Retirement example, the Alabama State Teacher's Retirement Association is the annuitant (in effect the Retirement Association is paying off the contributions made by the retiree and the employer prior to retiring).

Discrete Level Term-Life Annuity Issued To (x):

We have considered several of this type with the word "life" removed. Those payable annually (of course in unit amount) will be dealt with below and the life-annuities payable m^{th}ly will be relegated to the homework.

The *n-year life annuity-due* issued to (x) is an annuity-due in unit amount payable as long as (x) is alive but not beyond n years. A *n-year life annuity-immediate* issued to (x) is an annuity-immediate, payable in unit amount for n years contingent upon (x)'s survival.

And so on; for each type of annuity, there is a corresponding life-annuity. Obviously, the lender will expect to see less money coming when payments are scheduled under a life-annuity compared to an annuity-certain. How do we value life-annuities? For example, how much must Alabama State Teacher's Retirement Associate have available at the time of retirement of (x) in order to pay for (x)'s retirement? This is a little tricky because we don't know how long a particular life aged x will live, and, we cannot come up short (or too long).

In valuing life-annuities, we are viewing the payments at present (so we are determining the present value), and, we are taking into account actuarial (mortality) data. Let's look at a single payment at time t in unit amount made contingent upon the survival of (x). For the sake of argument, let us suppose that ASTRA is making the payment to each of the ℓ_x retiree's aged x.

The company must pay ℓ_{x+t} dollars - one to each survivor. Put another way, they must pay $\frac{\ell_{x+t}}{\ell_x} = {}_tp_x$ to each of (x) (on average). Therefore, the present value of the money each (x) receives is $\nu^t \cdot_t p_x$ where ν is computed using an assumed effective rate (this rate is picked by ASTRA and may be picked larger than any rate of inflation in part to create profit and to be cautious).

Turning this around slightly, if a payment of 1 is to be made t years from now contingent upon the survival of (x), then (x) "expects" to pay $\nu^t \cdot_t p_x$ (present dollars) in that (x) has a ${}_tp_x$ chance of making the dollar payment. We call $\nu^t \cdot_t p_x$ the *actuarial present value* or APV (for short), of a dollar payed in t years provided (x) has survived, valued at the present-time. The notation for this is ${}_tE_x$.

Given a life-annuity, we can proceed to calculate the APV of all the payments. Just as the notation for present value of standard annuities adorns a in various ways to indicate the type of annuity, the symbolism for the APV of a life-annuity modifies a_x in an analogous manner to indicate the type of life-annuity being used. The APV's of the standard discrete, level, life-annuities in annual unit amount are given by

$$a_x, \ddot{a}_x, a_x^{(m)}, \ddot{a}_x^{(m)}, a_{x:\overline{n}|}, \ddot{a}_{x:\overline{n}|}, a_{x:\overline{n}|}^{(m)}, \ddot{a}_{x:\overline{n}|}^{(m)}.$$

The APV's for the standard, discrete, increasing and decreasing life-annuities are

$$(I\,a)_x, (I\,a)_{x:\overline{n}|}, (I\,\ddot{a})_x, (I\,\ddot{a})_{\overline{n}|}, (D\,a)_{x:\overline{n}|}, (D\,\ddot{a})_{x:\overline{n}|},$$

and the APV's of the standard continuous, life-annuities are given by

$$\overline{a}_x, \overline{a}_{x:\overline{n}|}, (\overline{I\,a})_x, (\overline{I\,a})_{x:\overline{n}|}, (\overline{D\,a})_{x:\overline{n}|}.$$

And of course there are the deferred, life-annuities like

$$_{k|}a_x, \quad _{k|}\ddot{a}_x, \quad _{k|}a_x^{(m)}, \quad _{k|}\ddot{a}_x^{(m)}, \quad _{k|}a_{x:\overline{n}|}, \quad _{k|}\ddot{a}_{x:\overline{n}|}, \quad _{k|}a_{x:\overline{n}|}^{(m)}, \quad _{k|}\ddot{a}_{x:\overline{n}|}^{(m)}.$$

4.2 Valuing Life-Annuities

We've seen that the APV of a dollar issued at time t is worth $_tE_x = \nu^t \, _tp_x$ at present. This leads to direct valuations for each of the APV symbols in the previous section. Since we are discounting individual payments back to the present (or current) time, with respect to interest and survival, this approach is called the *current payment technique.*

For example,

$$\ddot{a}_x = \Sigma_{k=0}^{\infty}\nu^k \cdot_k p_x,$$

$$a_{x:\overline{n}|} = \Sigma_{k=1}^{n}\nu^k \cdot_k p_x,$$

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \Sigma_{k=0}^{nm-1}\frac{1}{m}\nu^{\frac{k}{m}} \cdot_{\frac{k}{m}} p_x,$$

and

$$(\overline{I}\,\overline{a})_x = \int_0^{\infty} te^{-\delta t} \cdot_t p_x \mu(x+t)dt,$$

are all derived under the (straightforward) current payment technique.

There is an alternate way to value annuities called the *present value of benefits* technique. Here we are conceptually considering the stream of payments under the annuity as a benefit. What is the present value of benefits awarded to (x)? Obviously the answer depends upon how long (x) lives so we must give a conditional answer. In the discrete case, the present value of benefits is the sum of the present values of all payments made up to (and including the payment at the beginning of the) k^{th} year, on the condition that (x) survives k years but not $k+1$. In the case where payments are m^{th}ly, replace year by m^{th} of the year, and k by the whole number of m^{th}'s lived.

Examples provide the most clarity because then we can put a face on the annuity in question. For a_x, the present value of benefits technique yields the formula

$$a_x = \Sigma_{k=0}^{\infty} a_{\overline{k+1}|} \cdot_k p_x \, q_{x+k}.$$

(Recall that ${}_kp_x\, q_{x+k}$ is the probability that (x) lives k full years, then dies before year's end).

The present value of benefits approach also gives

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \Sigma_{k=0}^{nm-1} \ddot{a}^{(m)}_{\overline{\frac{k+1}{m}}|} \cdot {}_{\frac{k}{m}}\, p_x \, {}_{\frac{1}{m}}\, q_{x+\frac{k}{m}} + \ddot{a}^{(m)}_{\overline{n}|} \cdot {}_n\, p_x.$$

Here, we have stretched the notation a bit: $\ddot{a}_{\overline{t}|}$ can be (and is) defined to be $\frac{1-\nu^t}{d^{(m)}}$ for any value $t > 0$. So, $\ddot{a}^{(m)}_{\overline{\frac{k+1}{m}}|}$ is the present value of payments if $\frac{1}{m}$ per m^{th} with interest rate $\frac{\imath^{(m)}}{m}$ per m^{th}.

Exercises

4.1 Write out closed form sums for a_x, and $a_x^{(m)}$, using the current payment technique.

4.2 Write out closed form sums for $a_{x:\overline{n}|}^{(m)}$, and $\ddot{a}_x^{(m)}$ using the current payment technique.

4.3 Write out closed form sums for $\overline{a}_{x:\overline{n}|}$, and $\overline{a}_x$ using the current payment technique.

4.4 Write out formulas for the APV symbols in problem 3.1 using the present value of benefits approach.

4.5 Write out formulas for the APV symbols in problem 3.2 using the present value of benefits approach.

4.6 Write out formulas for the APV symbols in problem 3.3 using the present value of benefits approach.

4.7 Write out the current payment formula and the p.v. of benefits formula for ${}_{n|}\ddot{a}_x$ and ${}_{n|}a_{x:\overline{n}|}$.

4.8 Argue that $a_{x:\overline{n}|} = a_x - {}_{n|}a_x$ and $\ddot{a}_{x:\overline{n}|} = \ddot{a}_x - {}_{n|}\ddot{a}_x$

4.9 Derive the formula $1 + a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} + {}_nE_x$.

4.10 Use the Illustrative Life Table data to calculate the APV's of the following:

(a) A whole-life annuity issued to (25) in the amount of $125,000.

(b) A whole-life annuity issued to (65) in the amount of $125,000.

(c) A whole-life annuity in the amount of $100,000 issued to (30) which is deferred 15 years.

(d) A twenty-year term life in the amount of $50,000 issued to (45).

4.11 Verify the following recursion relationships:

1. $\ddot{a}_x = 1 + \nu\, p_x \ddot{a}_{x+1}$.
2. $\ddot{a}_{x:\overline{n}|} = 1 + \nu\, p_x \ddot{a}_{x+1;n-1\rceil}$.
3. $a_x = \nu\, p_x + \nu\, p_x a_{x+1}$.
4. $\overline{a}_x = \overline{a}_{x:1\rceil} + \nu\, p_x \overline{a}_{x+1}$.
5. $\overline{a}_{x:\overline{n}|} = \overline{a}_{x:1\rceil} + \nu\, p_x \overline{a}_{x+1:n-1\rceil}$.

Chapter 5

Benefits Paid Upon Decrement

5.1 Life Insurance

In this chapter we will examine the present value of benefits when \$1 is paid upon the death of (x), either at the end of the year, or m^{th} of the year, or at the instant of death depending upon the policy contract. The word *discrete*, when modifying an insurance, refers to benefits payable either at the end of the year of death or the end of the m^{th} of death. When the insurance is discrete, the premium payments are also discrete and are collected according to when death-benefits are paid (i.e., annually or m^{th}ly).

A *discrete whole-life insurance policy* with unit benefit is a legal commitment from an insurance carrier to provide the beneficiary of the insured, \$1 at the end of the year (or end of the m^{th}) of death regardless of when the insured dies. (Of course an "insurance-due" would be hard to pay, but even harder for the insured to acknowledge the benefit).

A *discrete term-life policy* issued to (x) is a policy such that if (x) expires within the term limit n, the beneficiary receives a fixed benefit at the end of the year (or end of the m^{th}) of death, and receives nothing if (x) survives at least n years.

Of course insurance policies can come in any form that the market will bear, but we are primarily interested in the standard types. The standard insurance policies are basically those that pay a benefit B at the end of the year, m^{th} of the year, or instant, of death, and may include a term-limit on benefit payments. There is one type of standard insurance that is a slight variation, that being a term-life policy with the added benefit that the beneficiary also receives B in case the insured survives the term. This type of insurance is called an *endowment* insurance.

As you have gleaned from prior discussions, we are primarily interested in the cost of these insurance policies. That is, we wish to know the actuarial present value of benefits. Like the notation for life-annuities, the symbol for an APV of benefits is A_x dressed up to convey the type of insurance in force. However there is one change which allows us to accommodate the endowment insurance. There is precise notational rule that accounts for the new notation, but for now we will just learn the symbolism.

The *standard insurance policies offer unit benefit.* Analogous to the notation for life-annuities, the APV of the standard whole-life insurance is A_x. The term-life uses a slightly modified notational scheme; $A_{\overset{1}{x}:\overline{n}|}$ is the APV. The 1 above the x indicates that in order for benefits to be paid, (x) must expire *before* n. The endowment has APV $A_{x:\overline{n}|}$ - the absence of a 1 over either x or $\overline{n}|$ indicates that x or n may expire in any order to receive the benefit. Can you see that there is no distinction between $a_{\overset{1}{x}:\overline{n}|}$ and $a_{x:\overline{n}|}$? Can you make sense out of $A_{x:\overset{1}{\overline{n}|}}$?

Regarding the whole-life issued to (x) in unit amount, of the ℓ_x dollars to be handed out (one for each of the ℓ_x insured), $\ell_{x+k}\ q_{x+k}$ must be turned over at the end of the $k+1^{st}$ year. In effect, the company must collect $\ell_{x+k}\ q_{x+k}/\ell_x = \ {}_kp_x\ q_{x+k}$ from each of the ℓ_x policy holders in order to cover the bill at the end of the $k+1^{st}$ year. In present value dollars, the company must collect $\nu^{k+1}\ {}_kp_x\ q_{x+k}$ in order to pay benefits incurred during year k. Turning this around, if (x) wishes to provide his estate with \$1 at the end of the $k+1^{st}$ year provided death has occurred, they must be willing to pay $\nu^{k+1}_k p_x\ q_{x+k}$ at present.

Therefore,

$$A_x = \Sigma_{k=0}^{\infty} \nu^{k+1}\ \ {}_kp_x\ q_{x+k}.$$

In order to cover benefits, the insurance company must charge A_x at present to each of its policy holders; and conversely, to be insured in the amount of \$1 payable at the end of the year of death, (x) would need to pay A_x at present. More appropriately, if (x) wishes to benefit in the amount of P dollars at the end of the year of death (to pay for funeral and any dependent's educational costs, say), (x) must provide the insurance carrier with $P \cdot A_x$ presently. This excludes profits and costs, by the way.

For the term-life issued to (x), a similar reasoning applies but the payment of benefits is restricted by the term n. So,

$$A_{\overset{1}{x:\overline{n}|}} = \Sigma_{k=0}^{n-1} \nu^{k+1} \ {}_k p_x \ q_{x+k}.$$

The endowment adds a payment of \$1 at the end of n years contingent upon the survival of (x). Recalling that the actuarial present value of \$1 paid at the end of n years provided (x) survives was found to be ${}_nE_x = \nu^n \ {}_n p_x$, we obtain,

$$A_{x:\overline{n}|} = \Sigma_{k=0}^{n-1} \nu^{k+1} \ {}_k p_x \ q_{x+k} + \nu^n \ {}_n p_x = A_{\overset{1}{x:\overline{n}|}} +_n E_x.$$

Still other types of policies offer future benefits in the form of the annuities of the previous chapter, especially the deferred life-annuities. A typical retirement account on (x) is a deferred life-annuity (deferred $65 - x$ years perhaps), payable monthly in an amount P where P is determined by years-of-service, and a certain average of your annual salaries (for AU it's 2% per year of employment and the average is the average of your three best years of the last 1).

5.2 Annuities and Insurance

The APV of benefits obtained for life insurances are reminiscent of the formulas for life-annuities derived using the present value of benefits technique of Section 4.2. This leads to a relationship between the APV of benefits for a given insurance and the APV of the corresponding life-annuity.

For example, the discrete whole-life annuity issued to (x) has APV

$$a_x = \Sigma_{k=0}^{\infty} a_{\overline{k+1}\rceil} \ {}_kp_x \ q_{x+k} = \Sigma_{k=0}^{\infty} \frac{1-\nu^{k+1}}{\imath} \ {}_kp_x \ q_{x+k},$$

and since $\Sigma_{k=0}^{\infty} \ {}_kp_x \ q_{x+k} = 1$,

$$a_x = \frac{1-A_x}{\imath}, \text{ or } \ A_x + \imath a_x = 1.$$

For the term-life annuity-due, payable m^{th}ly in unit annual amount, the APV is

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \Sigma_{k=0}^{nm-1} \ddot{a}_{\overline{\frac{k+1}{m}}\rceil} \ {}_{\frac{k}{m}}p_x \ {}_{\frac{1}{m}}q_{x+\frac{k}{m}} + \ddot{a}_{\overline{n}\rceil} \ {}_np_x.$$

Here we have $\ddot{a}_{\overline{\frac{k+1}{m}}\rceil} = \frac{1-\nu^{\frac{k+1}{m}}}{d^{(m)}}$ and $\Sigma_{k=0}^{nm-1} \ {}_{\frac{k}{m}}p_x \ {}_{\frac{1}{m}}q_{x+\frac{k}{m}} = {}_n \ q_x$, so,

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{{}_nq_x - A^{(m)}_{\overset{1}{x}:\overline{n}\rceil}}{d^{(m)}} + \frac{1-\nu^n}{d^{(m)}} \ {}_np_x = \frac{{}_nq_x + \ {}_np_x - A^{(m)}_{\overset{1}{x}:\overline{n}\rceil} - \nu^n \ {}_np_x}{d^{(m)}}.$$

Thus,

$$A^{(m)}_{x:\overline{n}\rceil} + d^{(m)} \ddot{a}^{(m)}_{x:\overline{n}|} = 1.$$

Convince yourselves that the following formulas are correct:

$$A^{(m)}_{x:\overline{n}\rceil} + \imath^{(m)} a_{x:\overline{n}|} = 1.$$

$$\imath^{(m)} a_{x:\overline{n}|} = d^{(m)} \ddot{a}^{(m)}_{x:\overline{n}|},$$

$$A_x = A_{\overset{1}{x}:\overline{n}|} + \ {}_nA_x,$$

$$A_x + d a_x = 1,$$

$$A_{x:\overline{n}|} = \nu \ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}\rceil},$$

$$a_{x:\overline{n}|} = \ {}_1E_x \ a_{x+1:\overline{n}\rceil},$$

$${}_{n|}a_x = \frac{A_{x:\overline{n}|} - A_x}{d} - \ {}_nE_x.$$

Exercises

5.1 Come up with a formula for $A_x^{(m)}$.

5.2 Assuming that $\ell_x = 100 - x$ for $0 \leq x \leq 100$, and that $\imath = .05$, calculate $A_{40:\overline{20}}$ and $(IA)_{40}$.

5.3 Explain what is meant by $A_{\overset{1}{x}:\overline{n}\rceil}$ and give an alternate expression for this.

5.4 If $A_x = .25, A_{x+20} = .4$, and $A_{x:\overline{20}\rceil} = .55$, calculate $A_{\overset{1}{x}:\overline{20}\rceil}$.

5.5 Derive the equations $A_x + d\ddot{a}_x = 1$, and $A_x^{(m)} + i^{(m)} a_x^{(m)} = 1$.

5.6 Use the Illustrative Life Table data to calculate the APV's of the following:

(a) A whole-life insurance issued to (27) in the amount of \$125,000.

(b) A whole-life insurance issued to (67) in the amount of \$125,000.

(c) A whole-life insurance in the amount of \$100,000 issued to (30) which is deferred 20 years.

(d) A twenty-year term life insurance in the amount of \$75,000 issued to (45).

5.7 Obtain a formula comparing $a_{x:\overline{n}|}$ and $A_{\overset{1}{x}:\overline{n}|}$.

5.8 Derive the formula $A_{x:\overline{n}\rceil}^{(m)} + \imath^{(m)} a_{x:\overline{n}|} = 1$.

Chapter 6

Paying for Insurance with Life-Annuities

6.1 Premiums

Suppose (x) wishes to buy a whole-life policy. If $x = 30$, and we use the Illustrative Life Table I, a $\$100,000$ policy (a typical benefit) will cost roughly $\$10,248.35$. Maybe (30) doesn't have $\$10,248.35$, or, seeing this lump sum figure, rethinks the need for insurance altogether. These are some of the reasons that companies allow policy holders to make payments, that hopefully appear to be small enough to overcome any resistance to buying insurance. Obviously, the payments should cease when the insured dies.

Given an insurance policy, the *net single premium* or *single benefit payment*, SBP, is the actuarial present value of benefits of the policy. The net single premium is the amount in present dollars that the insurance carrier must pay the average insured, and so this amount must be collected as a single premium or payment at present to offset costs. The net single premium does not account for profit as reflected in the qualifier "net". Premiums that account for net premiums, profit and adverse loss reaction are called *loaded* and are considered much later.

The standard approach is to use a life-annuity with level payments to pay for an insurance policy. Again we will only consider *net premiums*, the life-annuity payments that cover the cost (from an actuarial standpoint) of insurance only.

6.2 Annual Premiums For Annual Insurance

The generic symbol P represents the premium payable according to the particular life-annuity schedule in order to pay off an insurance. Let us understand how the notation is laid out. For our standard annuities, P will be a level annual payment, payable either m^{th}ly or continuously.

To convey the insurance type with our symbolism we place the symbol for the APV of benefits of the insurance policy inside the parenthesis in $P(\cdot)$. For example, $P(A_x)$ represents the premiums when paying for a whole-life issued to (x), while $P({}_{k|}\ddot{a}_x^{(m)})$ is the premium payment in the series of payments that pays for a deferred whole-life annuity-due, payable m^{th}ly (the benefit could be retirement income in this case).

The symbolism is incomplete so far as it doesn't tell us what the type of annuity it is that we are using. We dress $P(\cdot)$ up on the outside of the right parenthesis to convey the annuity type we are using to pay for the insurance. A stipulation is that the life-annuities that pay for insurance *are always of the annuity-due* variety. This is because any other annuity type would lead to the situation where the carrier must pay benefits without having ever received a premium.

Continuous annuities and immediate benefit-payments are quite practical. In this case we refer to the premiums as being *fully continuous*. When the premium payments are discrete but benefit-payments are at the moment of death, we refer to the premium as *semi-continuous*. Then there is the *fully-discrete* format.

An example of a fully-discrete premium is the level premium on a discrete whole-life issued to (x): $P(A_x)_{x:\overline{n}|}$ is the premium payment made at the beginning of each year for n years provided (x) is alive, in order to pay for a whole-life issued to (x). The premiums for an endowment payable m^{th}ly is $P(A_{x:\overline{n}|}^{(m)})_{x:\overline{n}|}^{(m)}$.

When the symbolism for the insurance coincides with the symbolism for the life-annuity, we can shorten the notation for the annuity-premium. For example, the net-annual premium for an endowment payable m^{th}ly is written as $P^{(m)}_{x:\overline{n}|}$. If the term h for premium payments is less than the term for benefit payments, the premium-term h is written in the lower-left corner of P. For example, an n-year endowment payable at the instant of death, with premium payments made continuously for h years has annual premium

$$_h\overline{P}_{x:\overline{n}|}.$$

When the term on the premium-annuity is not given, it is assumed to be the term for insurance benefits.

6.3 Determining Premiums

It is very easy to determine what the net level premiums must be in order to pay for insurance under a given annuity plan. If P is the level premium, and a is the APV of payments of \$1 under the annuity schedule, then Pa must equal A, where A is the net single premium (cost of insurance without loading); i.e., the present value of benefits. So,

$$P = \frac{A}{a}.$$

The premiums for a discrete endowment payable annually are

$$P_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{a_{x:\overline{n}|}}.$$

The premiums for a whole-life insurnce, payable at the moment of death, with premiums paid continuously for h years is

$$_h\overline{P}_x = \frac{\overline{A}_x}{\overline{a}_{x:\overline{n}|}}.$$

The fully-discrete premium for a term-life payable at the end of the m^{th} of the year of death, with level premiums paid every m^{th} is

$$P^{(m)}_{\overset{1}{x}:\overline{n}|} = P(A^{(m)}_{\overset{1}{x}:\overline{n}|})^{(m)}_{x:\overline{n}|} = \frac{A^{(m)}_{\overset{1}{x}:\overline{n}|}}{\ddot{a}^{(m)}_{x:\overline{n}|}},$$

while the premiums for an n-year endowment payable at the end of the m^{th} of death, with premiums payable for h years is

$$ {}_hP^{(m)}_{x:\overline{n}|} = \frac{A^{(m)}_{x:\overline{n}|}}{\ddot{a}^{(m)}_{x:\overline{h}|}}. $$

Here is an example.

Problem: Determine the net-annual premium used to purchase an n-year term insurance whose benefit calls for the return of all premiums paid up to the time of death plus a fixed benefit of $100,000.

Solution: Let P be the net annual premium for this policy. Then, the APV of benefits is

$$ \Sigma_{k=0}^{n-1}(100,000 + (k+1)P)\nu^{k+1}\ {}_kp_x\ q_{x+k} = 100,000A_{\overset{1}{x}:\overline{n}|} + P\cdot(IA)_{\overset{1}{x}:\overline{n}|}. $$

So, P is determined by the equation

$$ P\ddot{a}_{x:\overline{n}|} = 100,000A_{\overset{1}{x}:\overline{n}|} + P(IA)_{\overset{1}{x}:\overline{n}|}, $$

and

$$ P = \frac{100,000A_{\overset{1}{x}:\overline{n}|}}{\ddot{a}_{x:\overline{n}|} - (IA)_{\overset{1}{x}:\overline{n}|}}. $$

Exercises

6.1 Write down the signature for a level premium, payable annually for m years, paying for a k-year deferred whole-life issued to (x).

6.2 Write down the notation for a level premium, payable monthly for an n-year term-life insurance.

6.3 Express in words what is meant by $P(\overline{A}_{\overset{1}{x}:\overline{n}|})^{(12)}_{x:\overline{n}|}$.

6.4 Express in words what is meant by $\overline{P}_{x:\overline{n}|}$.

6.5 Express in words what is meant by ${}_{k|h}P_x$.

6.6 Use the Illustrative Life Table data to determine the following net, level premiums:

(a) ${}_{20|}P_{35}$.

(b) $P^{\;1}_{30:\overline{30|}}$.

(c) P_{21}.

(d) $P_{47:\overline{20|}}$.

6.7 Suppose (x) wishes to purchase a modified endowment. The benefit of \$100,000 is payed at the end of the year of death provided (x) dies within n years, and otherwise, (x) receives the return of all premiums at the end of n years if (x) survives until then. If a level, life annuity-due, payable for n years is used to pay for this insurance, calculate the premium.

6.8 Rank in increasing order of size:

i) $P(\overline{A}_x)$.

ii) $\overline{P}_x$.

iii) $P^{(4)}(\overline{A}_x)$.

iv) $P^{(2)}(\overline{A}_x)$.

Chapter 7

Uniform Distributions of Death and a Constant Force of Mortality

7.1 UDD and CF

Recall that uniform distribution of deaths, UDD, is an assumption on mortality which linearly interpolates the death-rate between integral ages to obtain the rates for fractional ages. This works out to

$$ {}_tq_x = t\ q_x \quad \text{for} \quad) < t < 1, \ \ x \ \text{ an integer.} $$

In this case,

$$ {}_tp_x = 1 - t\ q_x \qquad \text{and} \qquad {}_tp_x\mu(x+t) = \ q_x, $$

for x integral and $0 < t < 1$.

The constant force assumption, CF, on mortality dictates that the force of mortality $\mu(y)$ is constant between integral values of y. That is,

$$ \mu(x+t) = \mu(x), \quad \forall \quad x \in \mathbb{N}, \ 0 < t < 1. $$

We've seen that

$$ {}_tq_x = 1 - \ p_x^t $$

and

$$\mu(x+t) = -ln\ p_x,$$

so that

$${}_tp_x\mu(x+t) = -p_x^t\ ln\ p_x,$$

under this assumption.

7.2 Computations Assuming UDD or CF

We will produce some simplifying formulas when either UDD of CF are in force. Starting with a whole-life insurance, payable at the instant of death, we first make a general reduction which should serve us later:

$$\overline{A}_x = \int_0^\infty e^{-\delta t}\ {}_tp_x\mu(x+t)dt = \Sigma_{k=0}^\infty \int_0^1 e^{-\delta(k+t)}\ {}_{k+t}p_x\mu(x+k+t)dt,$$

and so,

$$\overline{A}_x = \Sigma_{k=0}^\infty e^{-\delta k}\ {}_kp_x \int_0^1 e^{-\delta t}\ {}_tp_{x+k}\ \mu(x+k+t)\ dt.$$

Under UDD, ${}_tp_{x+k}\ \mu(x+k+t) =\ q_{x+k}$ for $0 < t < 1$, and so, under UDD,

$$\overline{A}_x = \frac{1-e^{-\delta}}{\delta}\Sigma_{k=0}^\infty e^{-\delta k}\ {}_kp_x\ q_{x+k} = \frac{e^\delta - 1}{\delta}\Sigma_{k=0}^\infty e^{-\delta}e^{-\delta k}\ {}_kp_x\ q_{x+k},$$

and so,

$$\overline{A}_x = \frac{\imath}{\delta}\Sigma_{k=0}^\infty e^{-\delta(k+1)}\ {}_kp_x\ q_{x+k} = \frac{\imath}{\delta}A_x.$$

Assuming the force of mortality is constant throughout all ages (i.e., $\mu(x+t) = \mu$ for all x and t), then

$$\overline{A}_x = \int_0^\infty e^{-\delta t}\ {}_tp_x\ \mu(x+t)dt = \int_0^\infty e^{-\delta t}e^{-\mu t}\mu dt = \frac{\mu}{\mu+\delta}.$$

Under UDD again,

$$\overline{A}_{\overset{1}{x}:\overline{n}|} = \frac{\imath}{\delta}A_{\overset{1}{x}:\overline{n}|},$$

by Homework problem 7.3. Hence,

$$P(\overline{A}_{\overset{1}{x}:\overline{n}|}) = \frac{\overline{A}_{\overset{1}{x}:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = \frac{\frac{\imath}{\delta}A_{\overset{1}{x}:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}},$$

and

$$P(\overline{A}_{\overset{1}{x}:\overline{n}|}) = \frac{\imath}{\delta}P_x.$$

Exercises

7.1 Show that under UDD, $A_x^{(m)} = \frac{\imath}{\imath^{(m)}}A_x$.

7.2 Assume that the force of mortality is constant throughout all ages. Show that $A_{\overset{1}{x}:\overline{n}|} = A_x(1 - {}_nE_x)$.

7.3 Show under UDD that $\overline{A}_{x:\overline{n}|} = \frac{\imath}{\delta}A_{\overset{1}{x}:\overline{n}|} + {}_nE_x)$.

7.4 Assume that the force of mortality is constant throughout all ages. Show that $\overline{a}_x = \frac{1}{\mu+\delta}$.

7.5 Show that under UDD, $P(\overline{A}_x) = \frac{\imath}{\delta}P_x$.

7.6 Assume that the force of mortality is constant throughout all ages. Show that $\overline{P}(\overline{A}_x) = \mu = \overline{P}(\overline{A}_{\overset{1}{x}:\overline{n}|})$.

Chapter 8

Actuarial Loss and Benefit Reserve

8.1 Loss

Suppose an insured has a policy that has been in force for t years. From a retrospective point of view, some premium payments have been made, and since the insured is living, no benefits have been paid. The value of all *premium* payments made in years 1 to t, accumulated to time t, must be held in *reserve* to partially off-set future claims (there may be other premium payments to be made in later years to help pay the benefit).

From the prospective point of view, there is a benefit to be paid and a balance of premium payments to be collected. The former is viewed as a *loss* to the insurance carrier, but this loss is partially off-set by pending premium payments (if any); the company anticipates a loss, the extent of the loss of course depends upon survival (longer life lessens the severity of the loss).

The future life-time of (x) is uncertain, rather we can only give the probability that (x) expires within a certain number of years: $T(x)$ is called the *future-lifetime random variable* for (x); by this we mean, the probability that $T(x)$ assumes values $t \geq a$ is just ${}_a p_x$. The *curtate-future-life-time* random variable $K(x)$ similarly is defined so that the probability that $K(x) = k$ (i.e., that (x) lives k full years but not $k+1$) is ${}_kp_x\ q_{x+k}$.

The *loss* at time t, regarding a policy issued to (x), or *loss random variable*, is generally denoted by ${}_tL$, and will be a function of $T(x)$ or $K(x)$ depending upon whether we are handling the continuous case of the discrete case. There is no special adornment of symbols because what we are mainly concerned with is the *expected loss.* Note that the loss is examined at time t under the tacit assumption that (x) has survived to time t. Again we will examine ${}_tL$ by looking at different examples.

Suppose that (x) has taken out a discrete whole life-insurance policy in unit amount and that level payments are made, on an annual basis, as long as (x) survives. Recall that the premium payments are in the amount

$$P_x = \frac{A_x}{\ddot{a}_x},$$

at the beginning of each year. At the end of the k^{th} year (the instant payments for year $k+1$ begins), the loss random variable,

$${}_tL = \nu^{K-k} - P_x\ \ddot{a}_{\overline{K-k}|}.$$

When $k = 0$ the formula holds but of course the loss at time zero better be 0 (you can go into the insurance business with a positive-loss outlook). Again, this is a net loss.

Suppose our life aged x has been making premium payments on his/her n-year endowment in unit amount, scheduled with level payments for $h < n$ years. The premium payment is $P = {}_hP_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{h}|}}$, and the loss r.v. in this case depends upon the intervals in which K, and k lie:

$${}_tL = \nu^{K-k} - P\ddot{a}_{x+k:\overline{K-k}|}, \quad \text{for } 0 \le k < K < h,$$

$${}_tL = \nu^{K-k} - P\ddot{a}_{x+k:\overline{h-k}|}, \quad \text{for } 0 \le k < h \le K < n,$$

$${}_tL = \nu^{K-k}, \quad \text{for } h \le k < K < n,$$

$${}_tL = \nu^{n-k}, \quad \text{for } k < n < K,$$

and

$${}_tL = 0, \quad \text{for } n \le k.$$

For a general policy, the APV of net-premium payments must be equal to the net single premium or *single benefit payment*, SBP in [4]. That the APV of net-premium payments is equal to the APV of benefits is not dependent upon which time we choose as the evaluation point. That is, at time t,

$$|—|—|— \cdots |—|—|— \cdots —|$$

looking in the prospective direction, we can compute the actuarial value of future losses. Retrospectively, we view a series of prior premium payments that we may accumulate (with mortality) to time t. The actuarially accumulated value of premium payments made in years 1 through t must equal the APV of future losses.

The sum of the actuarially accumulated values of net-premium payments made prior to the start of year $t+1$ is called the *net-premium reserve at time t*. Please note: Until it is necessary to distinguish between net premiums and *loaded* premiums, the words *premium* and *net premium* are synonymous. The qualifier "net" will sometimes be placed there for emphasis.

8.2 Net Premium Reserves

The *Equivalence Principle* asserts that our retrospective experience must equal our future expectations at any time t. From this principle, the net premium reserve is equal to the APV of future losses at time t. Some books define the premium reserve as the expected future loss, however I thought the definition should be consistent with the name.

Conceptually, the net-premium reserve is just the sum of all loses multiplied by the probability that the particular loss is attained. For some reason, the generic net premium reserve symbol at time t is ${}_tV$. You guessed it, we dress ${}_tV$ up to convey the type of insurance and the type of level-annuity schedule we are using.

The net-premium reserve at time k for a fully discrete whole-life issued to (x) is denoted by ${}_kV_x$. By the observations made thus far,

$$ {}_kV_x = \Sigma_{j=0}^{\infty}(\nu^{(k+j+1)-k} - P_x\ \ddot{a}_{\overline{(k+j+1)-k}|}) \cdot\ {}_jp_{x+k}\ q_{x+k+j} $$

$$ = \Sigma_{j=0}^{\infty}(\nu^{j+1} - P_x\ \ddot{a}_{\overline{j+1}|}) \cdot\ {}_jp_{x+k}\ q_{x+k+j}. $$

Perhaps you see that you do not need the actual loss r.v. to determine net-premium reserves. Consider the fully discrete n-year endowment, payable for h years with level payments.

$$ |—|—|— \cdots |—|—|— \cdots —| $$

At the end of k years, the net-premium reserve will be denoted by ${}_k^hV_{x:\overline{n}|}$. When $k < h$, this reserve can be found to be

$$ {}_k^hV_{x:\overline{n}|} = A_{x+k:\overline{n-k}|} - ({}_hP_{x:\overline{n}|})\ddot{a}_{x+k:\overline{h-k}|}. $$

When $k \geq h$, the reserve is

$$ {}_k^hV_{x:\overline{n}|} = A_{x+k:\overline{n-k}|}, $$

when the reserve is nonzero (i.e., $k < n$).

Exercises :

8.1 Derive the formula ${}_kV_x = 1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x}$.

Chapter 9

Multiple Decrement Models

9.1 Reasons for Decrement

When issuing a policy, the insured may wish to receive graded benefits depending upon the cause for decrement. For example, if the insurance provides a monetary benefit when the insured is no longer in their job (playing pro football for example), a larger benefit may be necessary if the cause for cessation is due to total physical incapacitation rather than natural retirement.

In general, we consider a description of mortality where ${}_tq_x^{(j)}$ is the probability that a life aged x decrements within t years due to cause j, where the various causes are numbered as $j = 1, 2, \ldots, k$. The symbol ${}_tp_x^{(j)}$ represents $1 - {}_tq_x^{(j)}$, and describes the probability that a life age x does not decrement within t years due to cause j, and of course accounts for those that decrement for some other cause and those that do not decrement within t years.

The probability that (x) decrements within t years due to any cause is denoted by ${}_tq_x^{(\tau)}$ and is computed as

$$ {}_tq_x^{(\tau)} = \Sigma_{j=1}^{k}\ {}_tq_x^{(j)}. $$

The probability that (x) does not decrement within t years is then ${}_tp_x^{(\tau)} = 1 - {}_tq_x^{(\tau)}$, which is of course generally smaller than $\Sigma_{j=1}^{k}\ {}_tp_x^{(j)}$. The expression

$$ {}_kp_x^{(\tau)}\ q_{x+k}^{(j)} $$

represents the probability that (x) dies within the $k+1^{st}$ year.

Notice, that in the multiple decrement model ${}_tq_x^{(j)}$, $j=1,2,\ldots,k$, there is an *associated single decrement model* ${}_tq_x^{(\tau)}$ and ${}_tp_x^{(\tau)}$.

Though we often consider discrete times for decrement, like during the k^{th} year, we assume that our probability data is continuous. Recall that the force of mortality is defined to convey the notion of (x) living t years, then (being hit with the force of mortality) suddenly decrementing. Now we must incorporate various causes for decrement.

The *force of mortality due to cause* j is defined to be

$$\mu_x^{(j)}(t) = \frac{d}{dt}\,{}_tq_x^{(j)} / \,{}_tp_x^{(\tau)};$$

so, figuratively, $\mu_x^{(j)}(t)\,{}_tp_x^{(\tau)}dt = \frac{d}{dt}\,{}_tq_x^{(j)}dt = {}_{t+dt}q_x^{(j)} - {}_tq_x^{(j)}$ connotes the probability that (x) lives t years then decrements within dt due to cause j. By the Fundamental Theorem of Calculus,

$$\int_0^t \mu_x^{(j)}(s)\,{}_sp_x^{(\tau)}ds = \int_0^t \frac{d}{ds}\,{}_sq_x^{(j)}ds = {}_tq_x^{(j)}.$$

The *total force of mortality* or *force of mortality due to all causes* is denoted by $\mu_x^{(\tau)}(t)$ and is taken to be

$$\mu_x^{(\tau)}(t) = \Sigma_{j=1}^k \mu_x^{(j)}(t).$$

We obtain the (consistent) expression

$${}_tq_x^{(\tau)} = \int_0^t \mu_x^{(\tau)}(s)\,{}_sp_x^{(\tau)}ds = \Sigma_{j=1}^k \int_0^t \mu_x^{(j)}(s)\,{}_sp_x^{(\tau)}ds = \Sigma_{j=1}^k\,{}_tq_x^{(j)}.$$

Observe that

$${}_\infty q_x^{(\tau)} = \int_0^\infty \mu_x^{(\tau)}(t)\,{}_tp_x^{(\tau)}dt = 1,$$

so that

$${}_tp_x^{(\tau)} = \int_t^\infty \mu_x^{(\tau)}(s)\,{}_sp_x^{(\tau)}ds.$$

As we've seen in single decrement models

$$ln\,{}_tp_x^{(\tau)} = \int_0^t \frac{d}{ds} ln\,{}_sp_x^{(\tau)}ds = \int_0^t -\mu_x^{(\tau)}(s)ds.$$

Therefore,

$$ {}_tp_x^{(\tau)} = e^{-\int_0^t \mu_x^{(\tau)}(s)ds}. $$

The entity ${}_tp_x^{(j)}$ does not fit into such an expression of this type, nor do we expect that it would. However we can define

$$ {}_tp_x^{'(j)} = \int_t^\infty \mu_x^{(j)}(s)\ {}_sp_x^{(\tau)}ds. $$

The corresponding probability data ${}_tp_x^{'(j)}$ for $j = 1, 2, \ldots, k$ represents the probability that (x) lives t years, then dies due to cause j at some later time. Of course,

$$ {}_tq_x^{'(j)} = 1 - {}_tp_x^{'(j)}. $$

Using the fact that $\mu_x^{(\tau)}(t)\Sigma_{j=1}^k \mu_x^{(j)}(t)$ we obtain that

$$ {}_tp_x^{(\tau)} = {}_tp_x^{'(1)} \cdot {}_tp_x^{'(2)} \cdots {}_tp_x^{'(k)} = \Pi_{j=1}^k\ {}_tp_x^{'(j)}. $$

A *probability density function* for a single decrement model ${}_tq_x$ is a function $f(s)$ for $s \geq 0$ such that

$$ {}_tq_x = \int_0^t f(s)ds. $$

In the single-decrement model ${}_tq_x^{(\tau)}$, a probability density function is $f(s) = \mu_x^{(\tau)}(s)\ {}_sp_x^{(\tau)}$. The literature emphasizes the dependence upon (x) and so the symbolism $f_T(s)$ is used, where T is the future-life-time random variable for life aged x (i.e., values of $T(x)$ are associated with ceratin probabilities). So,

$$ f_T(s) = \mu_x^{(\tau)}(s)\ {}_sp_x^{(\tau)}. $$

One speaks of the *associated density functions corresponding to the cause j for decrement*, $f_T(s, j)$. We define

$$ f_T(s, j) = \mu_x^{(j)}(s)\ {}_sp_x^{(\tau)}, $$

so that

$$ {}_tq_x^{(j)} = \int_0^t f_T(s, j)ds, $$

and

$$_tp_x^{\prime(j)} = \int_t^\infty f_T(s,j)ds.$$

Furthermore,

$$f_T(s) = \Sigma_{j=1}^k f_T(s,j),$$

and therefore,

$$\int_0^\infty f_T(s)ds = 1 = \Sigma_{j=1}^k \int_0^\infty f_T(s,j)ds.$$

Unlike the single decrement case,

$$\int_0^\infty f_T(s)ds \neq 1$$

Exercises :

8.1

Chapter 10

Bonds, Securities, and Yield Rates

10.1 Yield Rate of an Investment

Of practical importance, and of relevance in the Society of Actuaries Exam FM, is a generalization of an annuity. We examine the context where payments at perhaps irregular times over the course of a time interval, and the individual payments may be either positive or negative. Intuitively, one can picture a scenario where money is being put away for expenditures at later times; a negative payment can be construed as a return from the fund and a positive payment, a contribution. The *yield rate* of the fund is an interest rate such that the present value of all contributions equals the present value of returns. Yield rates may not exist and may not be unique if they exist. However, in the case where deposits are made up until some fixed date, and then withdrawals begin, then a yield rate exists and is unique ([**?**], pages 134 and 135). Below, any change in the fund will be called a contribution (so contributions may be either positive or negative).

Suppose that C_t is the contribution to the fund at time t as t varies throughout all relevant times; assume A is the initial deposit made into the fund, and that B is the fund-balance at the end of the year. The present value of all changes to the fund is

$$A + \Sigma_t C_t \nu^t - B\nu,$$

and the future value is

$$A(1+\imath)+\Sigma_t C_t(1+\imath)^{1-t}-B.$$

A *yield* rate $\imath$ is any annual effective rate for which

$$A+\Sigma_t C_t \nu^t = B\nu,$$

and

$$A(1+\imath)+\Sigma_t C_t(1+\imath)^{1-t}=B.$$

Example A company deposits 1000 at the beginning of the first year, and 150 per year, each year after into perpetuity. In return, the company receives payments at the end of the year into perpetuity of 100 for the first year, increasing by 5% each subsequent year. Which of the following is the yield rate for this investment?

(a) 4.7%
(b) 5.7%
(c) 6.7%
(d) 7.7%
(e) 8.7%

Using present values, the yield rate $\imath$ satisfies the following equation:

$$1000+\Sigma_{k=1}^{\infty}150\nu^k=\Sigma_{k=1}^{\infty}100(1.05)^{k-1}\nu^k,$$

where $nu=(1+\imath)^{-1}$ as usual. So,

$$1000+150/\imath = 150\nu(\frac{1}{1-\nu}) = 100\nu\Sigma_{k=0}^{\infty}(1.05\nu)^k = 100\nu\frac{1}{1-1.05\nu} = 100\frac{1}{\imath-.05}.$$

So,

$$10+\frac{1.5}{\imath}=\frac{1}{\imath-.05} \quad\Rightarrow\quad 10(\imath-.05)\imath+1.5(\imath-.05)-\imath=0,$$

and consequently $\imath^2 = .05(.15)$, and $\imath \approx .087$.

Suppose that an investment is made bearing an annual effective rate $\imath$, and that the interest cannot reinvested at the rate $\imath$, but rather at an annual effective rate $\jmath$. Specifically, consider the following two scenarios:

(1.) *A one-time investment of P is made earning an annual effective rate $\imath$. Interest is paid at the end of the year and immediately reinvested at the annual rate $\jmath$.*

After n years, the accumulated value of the investment is

$$P + \imath P s_{\overline{n}|\jmath}.$$

(2.) *Annual payments of P are made at the beginning of the year and earn an annual effective rate $\imath$. The interest earned is immediately reinvested at the annual rate $\jmath$.*

The first annuity-payment into the account bearing interest $\jmath$, is $\imath P$, the second is $2\imath P$, the third $3\imath P$ and so on. The accumulated value of these payments n-years beyond the present is

$$(\imath P)(1+\jmath)^n + (2\imath P)(1+\jmath)^{n-1} + \cdots + (n\imath P) = \imath P(Is)_{\overline{n}|\jmath}.$$

So, the accumulated value of the fund in (2.) is

$$nP + \imath P(Is)_{\overline{n}|\jmath}.$$

It is natural to ask for each of the investment funds, what is the effective yield? As mentioned above, there is a unique effective yield rate in each case. Let us denote the annual effective yield rate by r.

Fund (1.): We must solve

$$P(1+r)^n = P + \imath P s_{\overline{n}|\jmath};$$

or

$$(1+r)^n = \imath s_{\overline{n}|\jmath}.$$

Of course,

$$r = \sqrt[n]{\imath s_{\overline{n}|\jmath}} - 1.$$

Fund (2.): In this case we must solve

$$\ddot{s}_{\overline{n}|r} = (1+r)\frac{(1+r)^n - 1}{r} = n + \imath(Is)_{\overline{n}|\jmath},$$

for the yield rate r.

Example: A bank offers 1-year, 3-year, and 5-year termed certificates of deposit bearing nominal annual interest rates under quarterly compounding of 4.00%, 5.00% and 5.65% respectively. The certificates mature at the end of their respective terms and money may not be withdrawn early. Assuming that a customer invests 10,000 initially and reinvests all money upon maturity of certificate, what is the largest yield that the customer can obtain if the money is to withdrawn at the end of 6 years?

The answer is $\imath$ such that $(1+\imath)^6 = (1+(.0565/4))^{20}(1+(.04/4))^4$, which results in the effective rate $\imath \approx 5.48\%$. If any money were invested into the 3-year, 5% nominal interest certificate, then after 3-years, the money would have to be reinvested in a 3-year certificate in order the maximize yield. The resulting yield in two 3-year certificates is 5.09% effective.

An effective yield rate $\imath$ satisfies:

$$A(1+\imath) + \Sigma_t C_t (1+\imath)^{1-t} = B.$$

We can isolate the interest earned, I, during the year easily in this equation:

$$I = \imath A + \Sigma_t C_t ((1+\imath)^{1-t} - 1).$$

Given I, it is an easy exercise to determine $\imath$ (when possible) using a some computer software, or a financial calculator.

We can solve for certain yield rates directly by hand. When the interest earned on the contribution C_t, $(1+\imath)^{1-t}-1$, is approximated using a simple interest variant

$$(1+\imath)^{1-t}-1 \;\approx\; (1-t)\imath,$$

we obtain an estimate

$$I \;\approx\; \imath(A+\Sigma_t C_t(1-t),$$

which provides the approximation

$$\imath \;\approx\; \frac{I}{A+\Sigma_t C_t}.$$

The expression $\frac{I}{A+\Sigma_t C_t}$ is called the *dollar-weighted* or *money-weighted* yield rate.

The dollar-weighted yield rate may not provide an adequate description of the fund when the contributions are not small relative to the initial investment. Another adaptation of the yield rate is called the *time-weighted* yield, which does not depend upon the amount of the contributions. We will review those now.

Assume that contributions occur at times $t_1, t_2, \ldots, t_m$ and let

$$C'_k = C_{t_k}, \text{ and } B'_k = \text{ fund balance at } t_k \text{ (just before } C'_k.$$

The effective rate $\jmath_k$ for the interval $[t_{k-1}, t_k]$ satisfies

$$1+\jmath_k = \frac{B'_k}{C'_{k-1}+B'_{k-1}},$$

and so

$$\jmath_k = \frac{B'_k}{C'_{k-1}+B'_{k-1}} - 1.$$

The effective rate $\imath$ such that

$$\imath = (1+\jmath_1)(1+\jmath_2)\cdots(1+\jmath_m)-1,$$

is called the *time-weighted* yield rate.

Another instance in which $\imath$ tractable, is when all contributions are mid-year. That is, the net contribution $C = \Sigma_t C_t$ is made at mid-year. The formula for the yield rate becomes

$$A(1+\imath) + C(1+\imath)^{1/2} = B;$$

which is a quadratic in $(1+\imath)^{1/2}$. No approximation is done however mid-year contributions are assumed.

10.2 Sinking Funds

When payments are made towards repaying an outstanding loan, a portion of the payment covers interest accrued, and the remainder is used to pay down the debt. For example, for an n-year annuity-immediate in unit amount with level payments, a portion of the first year's payment of 1 covers the first year's interest, namely $\imath a_{\overline{n}|}$, and the remainder, $1-\imath a_{\overline{n}|} = \nu^n$, goes to pay down the principal. So, the outstanding debt at the beginning of the second year is

$$a_{\overline{n}|} - \nu^n = a_{\overline{n-1}|}.$$

Likewise, the outstanding balance at the beginning of the m^{th} year is

$$a_{\overline{n-m+1}|}$$

and the interest portion of the m^{th} payment is $\imath a_{\overline{n-m+1}|}$, with the residual amount $1 - \imath a_{\overline{n-m+1}|} = \nu^{n-m+1}$ going to pay down principal.

The analysis is a loan in terms of scheduled interest and premium pay-down payments is called the it amortization of a loan. A similar analysis applies to the other level annuities.

A *sinking fund* is a reconfiguration of the standard amortization schedule. When a loan is to be repaid, the lender may ask that the loan be repaid in a lump sum at the end of the term. In the interim, service must be paid in periodic installments in the form of interest payments. Extra payments are also placed into a fund, called a sinking fund; the sinking fund will then accumulate to the amount of the original loan (remember that interest has already been paid).

Example: Suppose that a loan in amount L is repaid over n years, by means of a sinking fund, with level payments at the end of the year. If P is the level payment, then $\imath L$ must go to service the loan and $P - \imath L$ goes into the sinking fund. The driving formula is the accumulated fund balance at the end of n years:

$$(P - \imath L)s_{\overline{n}|} = L,$$

from which we obtain

$$P = (\imath + \frac{1}{s_{\overline{n}|}})L.$$

If the interest rate afforded to the sinking fund is $\imath$, we find that the level payments under the Sinking Fund method are the same as in the standard method. By simple manipulation,

$$\frac{1}{a_{\overline{n}|}} = \frac{\imath}{1-\nu^n} = \frac{\imath(1+\imath)^n}{(1+\imath)^n - 1} = \frac{\imath}{(1+\imath)^n - 1} + \imath = \frac{1}{s_{\overline{n}|}} + \imath.$$

Therefore, the payment on a loan in amount L for an annuity-immediate is $P = L\frac{1}{a_{\overline{n}|}} = (\frac{1}{s_{\overline{n}|}} + \imath)L$ exactly as in the Sinking Fund method.

Suppose instead, the sinking fund only earns interest at an effective rate $\jmath$. The payment for the Sinking Fund method again is

$$P = (\imath + \frac{1}{s_{\overline{n}|\jmath}})L,$$

but now the interest born by the Sinking Fund is highlighted.

Since

$$s_{\overline{n}|\jmath} < s_{\overline{n}|\imath}$$

when $\imath > \jmath$, the factors involved in the Sinking Fund payments compare inversely:

$$\imath + \frac{1}{s_{\overline{n}|\jmath}} > \imath + \frac{1}{s_{\overline{n}|\imath}}.$$

Hence, a less favorable rate $\jmath$ in the sinking fund results in higher payments in the Sinking Fund method.

Exercises :

Reinvested Interest

10.1 An investor wishes to accumulate $5,000$ at the end of 10 years by making an initial deposit of P. The deposit earns 10% annual effective interest payed at the end of each year. The interest is immediately reinvested at an annual effective rate of 8%. Calculate P.

10.2 An investor wishes to accumulate $10,000$ at the end of 10 years by making level payments at the beginning of each year. The deposits earn an annual effective rate 12% per year payed at the end of the year. The interest is immediately reinvested at an annual effective rate of 8%. Calculate the level payment.

10.3 Payments of 200 are made at the end of each month for 5 years into an account bearing an 8% nominal interest rate. The interest is immediately reinvested into an account offering a 6% nominal rate of interest. Calculate the accumulated value of the fund at the end of 5 years.

Yield Rates

10.4 Calculate the yield rate for Exercise 10.1.

10.5 Calculate the yield rate for Exercise 10.2.

10.6 Calculate the yield rate for Exercise 10.3.

10.7 Mike receives cash flows of 100 today, 200 in one year, and 100 in two years. The present value of these payments under an annual effective rate $\imath$ is 364.46. Calculate $\imath$.

10.8 At a nominal rate of $\imath$ convertible semi-annually, an investment of 1000 immediately and 1500 at the end of the first year will accumulate to 2600 at the end of the second year. Determine $\imath$.

10.9 At the beginning of the year an investment fund is set up with an initial deposit of 1000. Another deposit of 1000 is made at the end of 4 months. Withdrawals of 200 and 500 were made at the end of 6 months and 8 months respectively. The amount in the fund at year's end if 1560. Calculate the dollar-weighted yield rate for this fund during the year.

10.10 At an annual effective rate $\imath$, the present value of a perpetuity-immediate starting with a payment of 200 and increasing by 5% each year thereafter is 46530. Calculate $\imath$.

10.11 The stock of company X sells for 75 a share assuming an annual effective rate $\imath$. Dividends are paid at the end of the year forever. The first dividend payment is 6, and subsequent year's dividends are 3% greater than the previous year's dividend. Calculate the yield $\imath$.

10.12 A company deposits 1000 at the beginning of the first year and 150 at the beginning of each subsequent year in perpetuity, into a fund. The company receives payments at the end of the year forever. The first payment is 100, and subsequent payments increase by 5% over the previous payment. Calculate the company's yield rate for this fund.

10.13 At the beginning of the year, a fund was established with an initial investment of 1000. Four months later another deposit of 1000 is made. Subsequent withdrawals of 200 and 500 are made at 6 months and 8 months respectively from the initial deposit. The amount at the end of the year is 1560. Calculate the dollar-weighted yield rate for the year.

10.14 Same information that was provided in number 12. Calculate the time-weighted yield rate for the year.

Amortization and Sinking Funds

10.15 Suppose that a loan of $10,000$, issued with an effective interest rate of 7% is paid by means of a sinking fund with level payments for 15 years paid at the end of the year. What are the payments into the Sinking Fund if the interest rate on the Sinking Fund is 5%?

10.16 Lori borrows $10,000$ for 10 years at an effective rate of 9%. At the end of each year she pays the interest on the loan and deposits the amount into a Sinking Fund necessary to repay the principal after 10 years. Calculate the total payments X assuming the Sinking Fund bears the interest rate 8%.

10.17 A loan is being repaid with 25 annual payments of 300 each. With the tenth payment an extra 1000 is paid, and then the balance is repaid over 10 with revised level annual payments. If the effective annual rate is 8%, determine the amount of each of the revised payments.

10.18 A bank customer takes out a loan of 500 with a nominal rate of 16% converted quarterly. The customer makes payments of 20 at the end of each quarter. Calculate the amount of principal in the fourth payment.

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