PROBLEMS IN GENERAL AND SET-THEORETIC TOPOLOGY

4/23/04

Below are problems posed at the Problem Session in General/Set-theoretic Topology at the 2004 Spring Topology and Dynamics Conference at the University of Alabama, Birmingham. Included with each problem are some remarks of the poser. Some problems sent to me by email by Justin Moore and Alan Dow are also included. Please note that the poser of a problem at the problem session is not necessarily the one who asked the question originally.

Problems by J. van Mill

Problem 1. *Is there a non-trivial zero-dimensional homogeneous subspace of the real line with the fixed-point property for homeomorphisms?*

Related to Problem 1 is:

Problem 2. *Is there a nontrivial compact zero-dimensional first countable homogeneous space with the fixed-point property for homeomorphisms?*

These questions are rather old and asked many times. I believe they are interesting since zero-dimensionality and the fixed-point property are more or less 'orthogonal' properties.

Problems by J. T. Moore

The solution to the basis problem for linear orders (i.e., PFA implies that there are 5 specific uncountable linear orders such that every uncountable linear order isomorphically contains one) is relevant to problems such as the conjecture (L) (no L spaces), (PFA) every perfect compacta is premetric of degree 2, (PFA) the uncountable regular spaces have a 3 element basis. Perhaps I am being a bit ambitious but I would file this in the "promising new directions" category. As for problems:

Problem 3. *Does Martin's Maximum imply that every compact space either contains an uncountable discrete set or continuously maps 2-1 onto a metric space?*

Problem 4. *Does Martin's Maximum imply that every hereditarily Lindelöf regular space is hereditarily separable?*

Problem 5. *Does Martin's Maximum imply that the uncountable regular spaces have a three element basis (more specifically, that every uncountable regular space contains a homeomorphic copy of either an uncountable discrete space, or an uncountable subspace of the real line or the Sorgenfrey line)?*

Problem 6. *Do any of the three previous conclusions have non-trivial consistency strength? In particular do they imply the existence of $O^{\#}$?*

So far PFA has been sufficient but I am starting to suspect that MM may also be relevant in studying these problems via the new techniques used in proving a five element basis for the uncountable linear orders from PFA. I have listed the first three problems in the order that I feel that they are most tractable (whatever that means).

Remark by G. Gruenhage: Whether or not the conclusion of Problem 3 is consistent with ZFC is originally due to David Fremlin. Whether or not the 3-element basis problem for topological spaces is consistent is originally due to myself; I have also noted that if a positive answer to the 3-element basis problem is consistent with PFA, this also would give a positive answer to Fremlin's problem.

Problem 7. *If X is compact and X^2 is T_5, must X be separable?*

Remark. $MA(\omega_1)$ implies "yes"; this could be a nice ZFC fact though.

Problems by A.V. Arhangel'skii

A "space" means a "Tychonoff space". A base $\mathcal{B}$ of a space X is said to be of *countable order*, if, for every $x \in X$ and for every strictly decreasing sequence $\eta = \{U_n : n \in \omega\}$ of elements of $\mathcal{B}$ containing x, η is a local base at x in X.

H.H. Wicke and J. Worrell have established the following remarkable property of bases of countable order: if a space X has a base of countable order locally, then it has it globally. In particular, every locally metrizable space has a base of countable order. Hence, ω_1 has a base of countable order.

Problem 8. *Is every linearly Lindelöf space X with a base of countable order Lindelöf?*

Note, that if the answer is "yes", then every such X is separable and metrizable, since every paracompact space with a base of countable order is metrizable (Arhangel'skii, 1963). Note also that every locally metrizable linearly Lindelöf space is separable metrizable (Arhangel'skii and Buzyakova, 1999).

Problem 9. *Is the product of two (of arbitrary countable family) of linearly Lindelöf p-spaces linearly Lindelöf?*

The product of any countable family of Lindelöf p-spaces is Lindelöf, since every Lindelöf p-space admits a perfect mapping onto a separable metrizable space (Arhangel'skii, 1963). However, not every linearly Lindelöf locally compact space is Lindelöf, as it was shown by K. Kunen at this conference.

On the other hand, A. Karpov has proved that the product of any countable family of Čech-complete linearly Lindelöf spaces is linearly Lindelöf.

A space X is said to be *discretely Lindelöf* if the closure of every discrete subspace in X is Lindelöf. Every discretely Lindelöf space is linearly Lindelöf.

Problem 10. *Is every discretely Lindelöf space Lindelöf?*

V.V. Tkachuk presented the above problem at the conference, though it is due originally to Arhangel'skii, who pointed out that the analogous statement about discrete compactness is true and was published by M. Katetov around 1950.

Problem 11. *Is every locally compact discretely Lindelöf space X Lindelöf?*

The last problem is especially interesting in connection with Kunen's result cited above.

Problem by J. Dijkstra

A space is called *almost zero-dimensional* if every point has a neighbourhood basis consisting of sets that can be written as intersections of clopen subsets of the space. A space is called *cohesive* if every point has a neighbourhood that contains no nonempty subset that is clopen in the space. If a space X is totally disconnected then the clopen subsets of the space serve as a basis for a zero-dimensional Tychonoff topology ζ_X on the space. Remark 4.8 in [Dijkstra and van Mill, *Homeomorphism groups of manifolds and Erdős space*, Electron. Res. Announc. Amer. Math. Soc. **10** (2004), 29–38] states that for any cohesive, almost zero-dimensional, separable metric space X the topology ζ_X has uncountable character at every point. The standard example of such a space is Erdős space $\mathfrak{E}$, which consists of the vectors in Hilbert space ℓ^2 that have only rational coordinates.

Problem 12. *Is the character (or weight) of $\zeta_{\mathfrak{E}}$ equal to $2^{\aleph_0}$?*

Problems by S. Popvassilev

Call a space X *base-base paracompact* (Ted Porter) if X has a base $\mathcal{B}$ such that every base $\mathcal{B}' \subseteq \mathcal{B}$ has a locally finite subcover $\mathcal{C} \subseteq \mathcal{B}'$.

Problem 13. *Is every subspace X of the Sorgenfrey line S base-base paracompact? Are the irrationals with the topology induced from S base-base paracompact? What about a Bernstein subspace of S?*

It is known that every F_σ subset of S is base-base paracompact, as is every Lusin subset, and, under MA, every subset of cardinality $< 2^{\aleph_0}$.

The above question is a special case of the following one asked by Ted Porter.

Problem 14. *Is every paracompact Hausdorff space base-base paracompact?*

Remark: Every base-base paracompact space is a D-space (Porter).

Related problem included by G. Gruenhage:

Problem 15. *Must every paracompact Hausdorff space X be base-paracompact? I.e., must there be a base $\mathcal{B}$ with $|\mathcal{B}| = w(X)$ such that every open cover of X has a locally finite refinement by members of $\mathcal{B}$?*

Remark: Base-paracompactness was introduced and studied by Ted Porter in Topology Appl. **128**, (2003), 145–156.

Problem by M. Henriksen

The problem that follows was posed to me by J.Nagata at the conference in Matsue, Japan.

Problem 16. *If X is a Tychonoff space, let $C(X)$ denote the set of all continuous real-valued functions defined on X. Characterize metrizability of X algebraically in case $C(X)$ is considered either as a ring, a lattice, or a multiplicative semigroup.*

Remarks:1) One should expect to assume that X is realcompact and hence that each of its closed discrete subspaces has nonmeasurable cardinality 2) The characterization should be internal involving only algebraic invariants of $C(X)$, but not those of any larger algebras.

Problems by P. J. Nyikos

Here is the first question I posed in the problem session, the way Jimmie Lawson told it to me last summer:

Problem 17. *Begin with the compact-open topology on the set of continuous functions from N^N to N, and take the sequential modification (in which a set is closed iff it is sequentially closed, i.e. contains all limits of sequences converging from it). Is the resulting space* 0*-dimensional?*

Related questions: Is it regular? If it is regular, is it 0-dimensional? Is its complete regularization (meaning: take the weak topology generated by continuous real-valued functions) 0-dimensional? All these questions are open; we do not even have consistency results for any of them.

It is however known that the two topologies mentioned in Problem 17 are not the same; in fact, 5.12 of the following paper gives an example of a clopen subspace in the sequential modification that is neither open nor closed in the compact-open topology.

"Comparing Cartesian-closed Categories of (Core) Compactly Generated Spaces, by M. Escardo, J. Lawson and A. Simpson, to appear in Top. Appl.; preprint available at Simpson's website, http://www.dcs.ed.ac.uk/home/als/

The first few paragraphs of the following preprint give a very readable informal account of what this has to do with computer science:

Dag Normann, "Comparing hierarchies of total functionals," preprint available from his webpage: http://www.math.uio.no/ dnormann/

It talks about two kinds of infinite languages for specifying real numbers, real-valued functions of a real variable, and so on up a hierarchial line; a nice example of a function on the next step in the hierarchy is the definite integral of a continuous function $f : \mathbb{R} \to \mathbb{R}$. The two approaches have been shown to be equivalent this far up the hierarchy but the very next step is still open; the two kinds of languages are equivalent at this step if and only if the answer to the last of the related problems is "Yes"; of course, that would follow from an affirmative answer to Problem 17.

A published account of some earlier work, already showing how equivalence follows from a "Yes" answer to Problem 17, is:

"Comparing Functional Paradigms for Exact Real-number Computation," by Alex Simpson, Andrej Bauer and Martin Escardo, in Proceedings ICALP 2002, Springer Lecture Notes in Computer Science 2380, pp. 488-500, 2002.) [Not to be confused with Springer's Lecture Notes in Mathematics.]

Simpson's website, given above, has this in pdf form.

Normann, in the preprint cited above, shows the equivalence continues all the way up the hierarchy as long as the higher analogues of the last related problem are true. For each step in the hierarchy there is a certain cosmic space for which the key question is whether its complete regularization is 0-dimensional.

Now for something completely different:

Problem 18. *Is it consistent that compact homogeneous T_5-spaces are first countable? Could this even be true in ZFC? Is it true in ZFC that they are of cardinality $\leq \mathfrak{c}$? [It is true in the model obtained by adding $2^{\aleph_1}$ Cohen reals to a model of ZFC. See reference below.]*

Related problems: Is every homogeneous compact space of countable tightness of cardinality $\leq \mathfrak{c}$? first countable? Arhangel'skii conjectured "yes" answers in:

"Topological homogeneity. Topological groups and their continuous maps," Uspekhi Mat. Nauk 42 (2)(254) (1987) 69–105, 287.

The Proper Forcing Axiom (PFA) implies "yes" answers to both. This is shown in "Cardinal restrictions on some homogeneous compacta," by Juhasz, Nyikos and Szentmiklossy, to appear in Proceedings AMS. An almost-final version can be found in my website, http://www.math.sc.edu/ nyikos/preprints.html

Problem by M. Elekes

Problem 19. *Characterize the possible order types of the linearly ordered subsets of $B_1[0,1]$, where $B_1[0,1]$ is the class of real-valued Baire class 1 functions defined on $[0,1]$, partially ordered under the natural pointwise ordering. (That is; $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$, so $f < g$ iff $f(x) \leq g(x)$ for all $x \in X$ and $f(x) \neq g(x)$ for at least one $x \in X$.)*

Remarks: The answer to the corresponding problem concerning $C[0,1]$ is easy, exactly the real order types are possible (that is; orders that are order isomorphic to some set of reals). For Borel functions, in fact for Baire class 2 or higher even the problem of well-ordered subsets is independent. A partial answer for Baire class 1 is that under MA the possible order types of cardinality less than 2^ω are exactly the ones not containing ω_1 or ω_1^*.

Problem by A. Shibakov

Problem 20. *Can there be a Fréchet topological group G and a compact Fréchet space X such $G \times X$ is not Fréchet?*

Such a group G could be chosen to be countable without loss of generality (if it exists).

Remark by G. Gruenhage: There are, in ZFC, two Fréchet groups whose product is not Fréchet, as well as two compact Fréchet spaces whose product is not Fréchet, but there seem to be no known counterexamples, even consistent ones, for this problem. And Peter Nyikos reminded me of the relevant information (noting that the group G would be non-metrizable) that it is an unsolved problem of Malykhin whether or not there is in ZFC a countable Fréchet non-metrizable group.

Problems by A. Dow

Problem 21. *Is every compact ccc extemally disconnected image of ω^* separable (under PFA?)*

Problem 22. *If a compact X has a closed G_δ subset mapping onto βN, must X map onto βN?*

Problem 23. *Suppose X is compact of countable tightness, does there exist a discrete subset D whose closure has full cardinality? In particular, can X be written as a $\mathfrak{c}$-fold union of closures of discrete sets?*

Remark. This is a special case of an old problem of Efimov.

Problem 24 (Scarborough-Stone). *Must the product of every family of regular sequentially compact spaces be countably compact?*

Remark by P. Nyikos: Actually, Scarborough and Stone did not include any separation axioms in the statement of their problem. Nyikos obtained a counterexample in the class of Hausdorff spaces.

Problem 25. *Is it consistent that countably compact first-countable separable regular spaces are compact?*

Remark by P. Nyikos: I have offered $1,000 for a solution to Problem 25, which originally is due to Franklin and Rajagopalan.

Problem 26. *If a normal space X is the countable union of open metrizable subspaces, must X be metrizable?*

Problem 27. *Is it consistent that (normal) first-countable ω_1-collectionwise Hausdorff spaces are collectionwise Hausdorff?*

Problem 28. *Is the normal Moore space conjecture consistent with $\mathfrak{c} = \omega_2$?*

Problem by O. Pavlov

Problem 29. *Is there a regular maximal space that is a P-space?*

Recall that X is maximal if X is dense-in-itself but no stronger topology is. A space answering the question, if it exists, would have cardinality that is Ulam measurable.

Problems by S. Davis

A symmetric on a set X is a function $d : X \times X \to [0, \infty)$ such that the following are true:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

A topological space X is *symmetrizable* if and only if there is a symmetric d on X such that a subset $U \subset X$ is open iff for each $x \in U$ there exists $\varepsilon_x > 0$ such that $B(x, \varepsilon_x) \subset U$.

As usual, the ball $B(x, \varepsilon_x) = \{y \in X : d(x, y) < \varepsilon_x\}$.

There are three old questions about symmetrizable spaces which remain open in spite of quite a large amount of work.

Problem 30. *Is every point of a regular symmetrizable space a G_δ -set?*

This is an old question of E. A. Michael and A. V. Arhangel'skii. There is a Hausdorff, non-regular, counterexample, obtained by taking a Tychonoff symmetrizable space with a closed set which is not a G_δ, and shrinking the closed set to a point (done by Gruenhage and Nyikos).

Problem 31. *Is there a symmetrizable Dowker space?*

If there is a symmetrizable Dowker space, then one can attach an additional point to it to obtain a regular symmetrizable space in which that new point is not a G_δ. Almost conversely, Stephenson has shown that if a point x of a Hausdorff symmetrizable space X is not a G_δ, then $X\setminus\{x\}$ is not countably metacompact.

Problem 32. *Is it consistent that there are no symmetrizable L-spaces?*

An old result of Nedev shows that there are no symmetrizable S-spaces. Shakhmatov has constructed a model which contains a symmetrizable L-space. Balogh, Burke and Davis have constructed, in ZFC, a Hausdorff (non-regular) symmetrizable space which is hereditarily Lindelöf and not separable.