Calculus III

In studying Multivariable Calculus, it is critical to **visualize the geometric objects** (curves, planes, vectors, solids) in 3 dimension space. Use your pencil and sheets, your computer and keyboard, and your thinking and imagination to work on them!

Review: Vectors and Geometry of Space

Note: We should know how to compute and interpret the dot products, cross products, equations of lines and planes.

1. The dot product of two vectors is a scalar:

$$\langle x_1, x_2, x_3\rangle \cdot \langle y_1, y_2, y_3\rangle = x_1y_1 + x_2y_2 + x_3y_3.$$

The dot product is interpreted by $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

Ex. $\langle -1, 7, 4\rangle \cdot \langle 3, 1, 2\rangle = (-1)\times 3 + 7\times 1 + 4\times 2 = 12$,
$(\mathbf{i}+2\mathbf{j}-3\mathbf{k})\cdot(2\mathbf{j}-\mathbf{k}) = 1(0) + 2(2) + (-3)(-1) = 7$.

Recall that: $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, $\mathbf{k} = \langle 0, 0, 1\rangle$.

- Basic properties (**Ex**):

$$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2, \quad \mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}, \quad \mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$$

- θ can be computed by $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.
  Ex. Find the angle between $\mathbf{a} = \langle 2, 2, -1\rangle$ and $\mathbf{b} = \langle 5, -3, 2\rangle$.
- Schwartz Inequality: $|\mathbf{a}\cdot\mathbf{b}| \le |\mathbf{a}||\mathbf{b}|$, with equation hold if and only if $\mathbf{a}$ and $\mathbf{b}$ are on the same line.
- Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal iff $\mathbf{a}\cdot\mathbf{b} = 0$.
  Ex. Show that $-\mathbf{i}+2\mathbf{j}+5\mathbf{k}$ and $3\mathbf{i}+4\mathbf{j}-\mathbf{k}$ are orthogonal.
- The scalar projection and vector projection of $\mathbf{b}$ onto $\mathbf{a}$.
  Scalar projection of $\mathbf{b}$ onto $\mathbf{a}$: $\text{comp}_{\mathbf{a}}\mathbf{b} = |\mathbf{b}|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$.
  Vector projection of $\mathbf{b}$ onto $\mathbf{a}$: $\text{proj}_{\mathbf{a}}\mathbf{b} = \text{comp}_{\mathbf{a}}\mathbf{b} \times \frac{\mathbf{a}}{|\mathbf{a}|}$.

2. The cross product of two vectors on $\mathbf{R}^3$ is a vector:

$$\langle x_1, x_2, x_3\rangle \times \langle y_1, y_2, y_3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

   Vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{R}^3$. The direction of $\mathbf{a}\times\mathbf{b}$ is determined by right hand rule; The length of $\mathbf{a}\times\mathbf{b}$ equals to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

   - $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$.
   - Basic properties (Ex):

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}, \qquad (c\mathbf{a})\times\mathbf{b} = c(\mathbf{a}\times b) = \mathbf{a}\times(c\mathbf{b}),$$
$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}, \qquad (\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$$

   - Vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel iff $\mathbf{a}\times\mathbf{b} = \mathbf{0}$. (For example, $\mathbf{a}\times\mathbf{a} = \mathbf{0}$)
     Ex. (Ex 3, 4, p.817)
     (a) Find a vector perpendicular to the plane that pass through the points $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$.
     (b) What is the area of the triangle ΔPQR?
   - The volume of the parallelepiped determined by $\mathbf{a},\mathbf{b},\mathbf{c}$ is the magnitude of their scalar triple product:

$$V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|.$$

     Ex. Show that $\mathbf{a} = \langle 1,4,-7\rangle$, $\mathbf{b} = \langle 2,-1,4\rangle$, and $\mathbf{c} = \langle 0,-9,18\rangle$ are coplanar.

3. Equations of lines and planes.

   (a) Equations of lines:
   - Vector equation: $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$.
   - Parametric equations: Suppose $\mathbf{r} = \langle x,y,z\rangle$ and $\mathbf{v} = \langle a,b,c\rangle$. Then $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$.
   - Symmetric equations: Solve t in the parametric equations:

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}.$$

Ex. Find the equations of the line through the points $(1, 3, 2)$ and $(-4, 3, 0)$.

(b) Equations of planes:

- Vector equation: $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$.
  The vector $\mathbf{n}$ is a **normal vector** of the plane, that is, a vector orthogonal to the plane.
- Suppose $\mathbf{r} = \langle x, y, z \rangle$, $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, and $\mathbf{n} = \langle a, b, c \rangle$. We get a scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

- Linear equation: Combining constant terms $d := -ax_0 - by_0 - cz_0$, we have $ax + by + cz + d = 0$.

Ex. Find equations of the plane that passes through the points $P(1, 3, 2)$, $Q(3, -1, 6)$, and $R(5, 2, 0)$.

Ex. Show that the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$