Chapter 1

Basic Concepts

Let F be an arbitrary field in this chapter. Recall that an algebraic structure (e.g. vector space, group, ring, field, module, algebra) is a set with some binary operations that satisfies some axioms.

1.1 Definitions and Examples

History and Motivation

The study of Lie theory began in the second half of 19th century, when Marius Sophus Lie (a Norwagian mathematician) was motivated by Galois's theory in solving polynomail equations, and tried to find a similar solution to partial differential equations. He developed a theory of continuous transformation groups that preserve certain geometric structures, i.e. Lie groups. It turns out that most properties on the global structure of a Lie group can be determined the local infinitestimal structure, i.e. the Lie algebra associate with the Lie group. **In brief, Lie group is about continuous symmetries. Lie algebra is about infinitesimal symmetries.**

A detailed biographies of Sophus Lie and the originate of Lie theory can be found at: http://www-history.mcs.st-andrews.ac.uk/Biographies/Lie.html. An introduction of early history can be found at: https://en.wikipedia.org/wiki/Lie_group

Text: "Introduction to Lie Algebras and Representation Theory" by James Humphrey. (On algebraic closed field of characteristic zero, clear, concise, self-contained.)

References:

1. Complex semisimple Lie algebras, by J.P. Serre, brief.
2. Lie algebras, by Nathan Jacobson, cover results over several types of fields (char p, char 0, algebraically closed).

1.1.1 Lie algebra, subalgebra, and linear Lie algebra

Def. *A* **Lie algebra** *over* F*, denoted by* $(L, [])$*, is an* F*-vector space* L *with a binary operator*

$$[] : L \times L \to L, \qquad (x, y) \mapsto [xy] \quad \textit{(or } [x, y]\textit{)}$$

that satisfies the following axioms:

1. *(Bilinearity)* $[]$ *is bilinear.*
2. *(Skew Symmetry)* $[xx] = 0$ *for all* $x \in L$*.*

3. *(Jacobi identity)* $[x[yz]] + [y[zx]] + [z[xy]] = 0$ *for all* $x, y, z \in L$.

$[]$ *is called the* **bracket** *or* **commutator**.

Remark. 1. $[xx] = 0 \Longrightarrow [xy] = -[yx]$. *Conversely, if* $char\,\mathrm{F} \neq 2$, *then* $[xy] = -[yx] \Longrightarrow [xx] = 0$.

2. *In general, Lie algebra is not associate:* $[x[yz]] \neq [[xy]z]$.

Ex. *Let* $V = \mathrm{F}^n$. *In* $End(V) \simeq \mathrm{F}^{n\times n}$, *define* $[X, Y] = XY - YX$. *Verify that* $(\mathrm{F}^{n\times n}, [])$ *is a Lie algebra. It is called the* **general linear algebra** *and denoted by* $\mathfrak{gl}(V) = \mathfrak{gl}(n, \mathrm{F})$.

Def. *A subspace* K *of* L *is called a (Lie)* **subalgebra** *if* $[xy] \in K$ *for all* $x, y \in K$, *denoted by* $K \leq L$.

Ex. *The following are Lie subalgebras of* $\mathfrak{gl}(n, \mathrm{F})$:

1. *the subspace* $\mathfrak{t}(n, \mathrm{F})$ *of upper triangular matrices in* $\mathfrak{gl}(n, \mathrm{F})$;
2. *the subspace* $\mathfrak{n}(n, \mathrm{F})$ *of strictly upper triangular matrices in* $\mathfrak{gl}(n, \mathrm{F})$;
3. *the subspace* $\mathfrak{d}(n, \mathrm{F})$ *of diagonal matrices in* $\mathfrak{gl}(n, \mathrm{F})$.

We have $\mathfrak{t} = \mathfrak{d} + \mathfrak{n}$, $[\mathfrak{t}\mathfrak{t}] = \mathfrak{n}$ *and* $[\mathfrak{t}\mathfrak{n}] = \mathfrak{n}$.

Remark. 1. *Any subalgebra of* $\mathfrak{gl}(n, \mathrm{F})$ *is called a* **linear Lie algebra** *or* **matrix Lie algebra**.

2. *Any Lie algebra* L *such that* $[xy] = 0$ *for all* $x, y \in L$ *is called an* **abelian Lie algebra**, *such as* $\mathfrak{d}(n, \mathrm{F})$.

Def. *Two Lie algebras* $(L, [])$ *and* $(L_1, []_1)$ *are* **isomorphic** *if there exists a vector space isomorphism* $\phi : L \to L_1$ *satisfying*

$$\phi([xy]) = [\phi(x)\phi(y)]_1 \qquad \text{for all } \ x, y \in L,$$

and then ϕ *is called an* **isomorphism**.

Like the other algebraic structures, Lie algebras are often classified up to isomorphism.

Ex. *The only 1-dim Lie algebra over* F *is the abelian Lie algebra. (why?)*

Ex. *The 2-dim Lie algebras* $L = span(x, y)$ *over* F *are:*

1. *Abelian Lie algebra:* $[xy] = 0$.
2. *Non-abelian Lie algebra: we may choose* x, y *such that* $[xy] = y$.

$$L \simeq \left\{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \mid x, y \in \mathrm{F} \right\}.$$

Ex (HW)**.** F^3 *with the cross-product is a 3-dim Lie algebra:*

$$[uv] = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Thm 1.1 (Ado-Iwasawa)**.** *Every Lie algebra is isomorphic to certain matrix Lie algebra (*$char\,\mathrm{F} = 0$*: Ado,* $char\,\mathrm{F} = p$*: Iwasawa.)*

1.1.2 Classical algebras

There are four families of **classical algebras**, denoted by $\mathrm{A}_\ell, \mathrm{B}_\ell, \mathrm{C}_\ell, \mathrm{D}_\ell$, which are the typical examples of simple Lie algebras.

Let $[n] := \{1, 2, \cdots, n\}$. Define the *Kronecker delta function*

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

In $\mathrm{F}^{n\times n}$, let e_{ij} be the matrix that has the only nonzero enetry 1 in the (i, j)-position.

Lem 1.2. *$\{e_{ij} \mid i, j \in [n]\}$ is a basis of $\mathfrak{gl}(n, \mathrm{F})$, and*

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}.$$

Lem 1.3. *Every $s \in \mathrm{F}^{n\times n}$ corresponds to a bilinear form on $V = \mathrm{F}^n$ by*

$$\langle u, v\rangle = u^t s v, \qquad u, v \in V.$$

Define

$$\begin{aligned} L_s &:= \{x \in \mathfrak{gl}(n, \mathrm{F}) \mid \langle xu, v\rangle + \langle u, xv\rangle = 0\} \\ &= \{x \in \mathrm{F}^{n\times n} \mid sx = -x^t s\}. \end{aligned}$$

Then L_s is a Lie subalgebra of $\mathfrak{gl}(n, \mathrm{F})$.

Remark. *Bilinear forms are classified by congruence relationship. Suppose $s, g \in F^{n\times n}$ and g is nonsingular. What is the relation between L_s and $L_{g^t sg}$?*

$$\begin{aligned} L_{g^t sg} &= \{x \in \mathrm{F}^{n\times n} \mid g^t sgx = -x^t g^t sg\} \\ &= \{x \in \mathrm{F}^{n\times n} \mid sgxg^{-1} = -(g^t)^{-1} x^t g^t s\} \\ &= \{x \in \mathrm{F}^{n\times n} \mid sgxg^{-1} = -(g^t)^{-1} x^t g^t s\} \\ &= \{x \in \mathrm{F}^{n\times n} \mid gxg^{-1} \in L_s\} \\ &= \{x \in \mathrm{F}^{n\times n} \mid x \in g^{-1} L_s g\} = g^{-1} L_s g. \end{aligned}$$

So Lie algebras L_s and $L_{g^t sg} = g^{-1} L_s g$ are (inner) isomorphic.

Now we can describe the classical algebras (up to isomorphisms):

A_ℓ: Let $V = \mathrm{F}^{\ell+1}$. Let $\mathfrak{sl}(V) = \mathfrak{sl}(\ell + 1, \mathrm{F})$ consist of the elements of $\mathfrak{gl}(V) = \mathfrak{gl}(n, \mathrm{F})$ having trace zero. The **trace** of a matrix x is the sum of its diagonal entries. For all $x, y \in \mathfrak{gl}(V)$ we have

$$\mathrm{Tr}\,([xy]) = \mathrm{Tr}\,(xy - yx) = \mathrm{Tr}\,(xy) - \mathrm{Tr}\,(yx) = 0.$$

So $\mathfrak{sl}(V) = \mathfrak{sl}(\ell + 1, \mathrm{F})$ is a subalgebra of $\mathfrak{gl}(V)$, called the **special linear algebra**.

C_ℓ: Let $V = \mathrm{F}^{2\ell}$, and $s = \begin{bmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{bmatrix}$. The **symplectic algebra** is

$$\mathfrak{s}p(V) = \mathfrak{s}p(2\ell, \mathrm{F}) := L_s = \{x \in \mathrm{F}^{2\ell\times 2\ell} \mid x^t s = -sx\}.$$

Explicitly, $x = \begin{bmatrix} m & n \\ p & q \end{bmatrix} \in \mathfrak{s}p(2\ell, \mathrm{F})$ for $m, n, p, q \in \mathfrak{gl}(\ell, \mathrm{F})$ iff

$$n^t = n, \qquad p^t = p, \qquad m^t = -q.$$

Clearly $\mathfrak{s}p(2\ell, \mathrm{F}) \leq \mathfrak{sl}(2\ell, \mathrm{F})$. A basis of $\mathfrak{s}p(2\ell, \mathrm{F})$ consists of:

(a) the diagonal matrices $e_{ii} - e_{\ell+i,\ell+i}, \quad i \in [\ell]$;
(b) $e_{ij} - e_{\ell+j,\ell+i}, \quad i, j \in [\ell],\ i \neq j$;
(c) $e_{i,\ell+i}, \quad i \in [\ell]$; and $e_{i,\ell+j} + e_{j,\ell+i}, \quad 1 \le i < j \le \ell$;
(d) $e_{\ell+i,i}, \quad i \in [\ell]$; and $e_{\ell+i,j} + e_{\ell+j,i}, \quad 1 \le j < i \le \ell$.

So $\dim \mathfrak{sp}(2\ell, \mathrm{F}) = 2\ell^2 + \ell$.

B_ℓ: Let $V = \mathrm{F}^{2\ell+1}$ and $s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{bmatrix}$. The **orthogonal algebra**

$$ø(V) = ø(2\ell+1, \mathrm{F}) := L_s.$$

If $x \in \mathfrak{gl}(2\ell+1, \mathrm{F})$ is partitioned in the same form as s, say $x = \begin{bmatrix} a & b_1 & b_2 \\ c_1 & m & n \\ c_2 & p & q \end{bmatrix}$, then $x \in ø(2\ell+1, \mathrm{F})$ iff

$$a = 0, \quad c_1 = -b_2^t, \quad c_2 = -b_1^t, \quad q = -m^t, \quad n^t = -n, \quad p^t = -p.$$

We have $ø(2\ell+1, \mathrm{F}) \le \mathfrak{sl}(2\ell+1, \mathrm{F})$. A basis of $ø(2\ell+1, \mathrm{F})$ consists of

(a) $e_{ii} - e_{\ell+i,\ell+i}, \quad 2 \le i \le \ell+1$;
(b) $e_{1,\ell+i+1} - e_{i+1,1}$ and $e_{1,i+1} - e_{\ell+i+1,1}, \quad i \in [\ell]$;
(c) $e_{i+1,j+1} - e_{\ell+j+1,\ell+i+1}, \quad i, j \in [\ell],\ i \neq j$;
(d) $e_{i+1,\ell+j+1} - e_{j+1,\ell+i+1}$, $1 \le i < j \le \ell$;
(e) $e_{i+\ell+1,j+1} - e_{j+\ell+1,i+1}$, $1 \le j < i \le \ell$.

Therefore, $\dim ø(2\ell+1, \mathrm{F}) = 2\ell^2 + \ell$.

D_ℓ: Let $V = \mathrm{F}^{2\ell}$ and $s = \begin{bmatrix} 0 & I_\ell \\ I_\ell & 0 \end{bmatrix}$. The **orthogonal algebra**

$$ø(V) = ø(2\ell, \mathrm{F}) := L_s.$$

A matrix $x = \begin{bmatrix} m & n \\ p & q \end{bmatrix} \in ø(2\ell, \mathrm{F})$ iff

$$q = -m^t, \quad n^t = -n, \quad p^t = -p.$$

Similarly $ø(2\ell, \mathrm{F}) \le \mathfrak{sl}(2\ell, \mathrm{F})$. (<u>exercise</u>)A basis of $ø(2\ell, \mathrm{F})$ consists of

(a) $e_{ii} - e_{\ell+i,\ell+i}, \quad i \in [\ell]$;
(b) $e_{i,j} - e_{\ell+j,\ell+i}, \quad i, j \in [\ell],\ i \neq j$;
(c) $e_{i,\ell+j} - e_{j,\ell+i}$, $1 \le i < j \le \ell$;
(d) $e_{i+\ell,j} - e_{j+\ell,i}$, $1 \le j < i \le \ell$.

Therefore, $\dim ø(2\ell, \mathrm{F}) = 2\ell^2 - \ell$.

As we noted early, any matrix from the same congruence class over F will produce an isomorphic Lie algebra.

1.1.3 Abstract Lie algebra

Suppose L has a basis $\{x_1, \cdots, x_n\}$. The Lie algebra structure of L can be determined by the bracket operation on the basis:

$$[x_i x_j] = \sum_{k=1}^{n} a_{ij}^k x_k,$$

where a_{ij}^k are called the **structure constants**. The axioms (L2) and (L3) imply that (exercise)

$$a_{ii}^k = 0, \qquad a_{ij}^k + a_{ji}^k = 0, \qquad \sum_k (a_{ij}^k a_{kl}^m + a_{jl}^k a_{kl}^m + a_{li}^k a_{kj}^m) = 0.$$

1.1.4 Lie algebras of derivations

Def. *An* F**-algebra** *is an* F*- vector space* $\mathcal{U}$ *together with a bilinear operation*

$$\mathcal{U} \times \mathcal{U} \to \mathcal{U}, \qquad (a, b) \mapsto ab.$$

A **derivation** *of* $\mathcal{U}$ *is a linear endomorphism* $\delta \in End(\mathcal{U})$ *satisfying*

$$\text{the product rule:} \qquad \delta(ab) = a\delta(b) + \delta(a)b.$$

Let $Der(\mathcal{U})$ *denote the set of all derivations of* $\mathcal{U}$*.*

Der $(\mathcal{U})$ is not an F-algebra in general. However, it is closed under $[\delta\delta'] = \delta\delta' - \delta'\delta$ (exercise).

Thm 1.4. $Der(\mathcal{U})$ *is a Lie subalgebra of* $\mathfrak{gl}(\mathcal{U})$.

Ex. *The derivation algebra of a Lie algebra* $(L, [])$ *is*

$$Der(L) = \{\delta \in End(L) \mid \delta([x, y]) = [\delta(x), y] + [x, \delta(y)] \text{ for all } x, y \in L\}.$$

The Jacobi identity on L can be written as (by $[xy] = -[yx]$):

$$[z[xy]] = [[zx]y] + [x[zy]].$$

Define $\operatorname{ad} z \in \operatorname{End}(L)$ for any $z \in L$ by

$$\operatorname{ad} z\ (x) := [z, x] \qquad \text{for all } x \in L.$$

Then

$$\operatorname{ad} z([x, y]) = [\operatorname{ad} z(x), y] + [x, \operatorname{ad} z(y)].$$

Therefore, $\operatorname{ad} z \in \operatorname{Der}(L)$ for all $z \in L$.

Def. *Every* $\operatorname{ad} z \in Der(L)$ *is called an* **inner derivation**. *All others are called* **outer derivations.**

Def. *The* **adjoint representation** *of* L *is defined by*

$$\operatorname{ad} : L \to End(L), \quad z \mapsto \operatorname{ad} z.$$

Lem 1.5. $\operatorname{ad} L := \{\operatorname{ad} x \mid x \in L\}$ *is a Lie subalgebra of* $Der(L)$*. Precisely,*

$$[\operatorname{ad} x, \operatorname{ad} y] = \operatorname{ad}[x, y].$$

We will see that $\operatorname{ad} L$ is indeed an *ideal* of $\operatorname{Der}(L)$.

1.1.5 Lie groups and Lie algebras

Lie algebras are closely related to linear algebraic groups and Lie groups. We consider $\mathrm{F} = \mathbf{R}$ or $\mathbf{C}$ here for simplicity.

Def. *A group G is called a* **complex Lie group**, *if G is also a complex* differential manifold *(locally like $\mathbf{C}^m$) such that the group operations*

$$\cdot : G \times G \to G \qquad \text{and} \quad {}^{-1} : G \to G$$

are differentiable functions. Similarly for **real Lie group**.

Thm 1.6. *Every (real or complex) Lie group G corresponds to a Lie algebra of the same dimension, namely the tangent space $L := T_1G$ at $1 \in G$, with the bracket operation*

$$[XY] := XY - YX \qquad \text{for} \quad X, Y \in T_1G.$$

Conversely, a Lie algebra L may correspond (as the tangent space at identity) to many nonisomorphic Lie groups; these Lie groups have the same universal covering space, which is the unique connected simple connected Lie group corresponding to L.

Ex. *The real Lie groups $G_1 := \{e^{\imath\theta} \mid \theta \in \mathbf{R}\}$ has the Lie algebra $\imath\mathbf{R}$, and $G_2 := \mathbf{R}^+$ have the Lie algebra $\mathbf{R}$, where $\imath\mathbf{R}$ and $\mathbf{R}$ are isomorphic.*

However, there exist one-to-one correspondences between the matrix Lie algebras and the connected matrix Lie groups.

Thm 1.7. *Suppose $\mathrm{F} = \mathbf{C}$ or $\mathbf{R}$. Then every matrix Lie algebra $L \leq \mathfrak{gl}(n, \mathrm{F})$ corresponds to a unique connected matrix Lie group G, such that $L = T_1G$ (here $1 \in G$ is I_n.) Explicitly, G is generated by all elements*

$$\exp X := I_n + \sum_{k=1}^{\infty} \frac{1}{k!} X^k \qquad \text{for} \qquad X \in L,$$

Ex. *(Classical groups):*

1. *Lie algebra $\mathfrak{gl}(n, \mathrm{F})$ corresponds to the matrix group $\mathrm{GL}(n, \mathrm{F})$ of all $n \times n$ invertible matrices.*
2. *Lie algebra $\mathfrak{sl}(n, \mathrm{F})$ of trace 0 matrices corresponds to the matrix group $\mathrm{SL}(n, \mathrm{F})$ of all $n \times n$ matrices of determinante 1.*
3. *Recall that Lie algebras of types B, C, D are defined by*

$$L = \{x \in \mathrm{F}^{n \times n} \mid x^t s = -sx\}$$

where

$$C_\ell : \qquad n = 2\ell, \quad s = \begin{bmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{bmatrix};$$

$$B_\ell : \qquad n = 2\ell + 1, \quad s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{bmatrix};$$

$$D_\ell : \qquad n = 2\ell, \quad s = \begin{bmatrix} 0 & I_\ell \\ I_\ell & 0 \end{bmatrix}.$$

For each L above, the corresponding connected matrix Lie group G is the identity component of

$$\{g \in \mathrm{GL}(n, \mathrm{F}) \mid g^t s = s g^{-1}\}$$

for the same s defining L (exercise).

We may use $\mathrm{GL}(V)$ to denote the group of invertible endomorphisms of a vector space V. Similarly for $\mathrm{SO}(V)$ and $\mathrm{Sp}(V)$.

Homework

1. Verify that F^3 with the cross-product $\times$ is a Lie algebra.

2. Verify that the matrix form of elements of D_ℓ (associate with $s = \begin{bmatrix} 0 & I_\ell \\ I_\ell & 0 \end{bmatrix}$) is $x = \begin{bmatrix} m & n \\ p & q \end{bmatrix}$ where $q = -m^t$, $n^t = -n$, and $p^t = -p$. Then verify the dimension of D_ℓ.

3. (1.1.10)

   (a) Show that A_1, B_1, C_1 are all isomorphic, and D_1 is the one dimensional Lie algebra;

   (b) show that B_2 is isomorphic to C_2;

   (c) show that D_3 is isomorphic to A_3;

   (d) what can you say about D_2?

4. (1.1.11) Verify that the commutator of two derivations of an F-algebra is again a derivation, whereas the ordinary product need not be.