1.2 Ideals and homomorphisms

1.2.1 Ideals

Def. *A subspace I of a Lie algebra L is called an* **ideal** *of L ($I \trianglelefteq L$) if*

$$x \in L,\ y \in I \quad \Longrightarrow \quad [x, y] \in I \quad \text{(equiv. } [y, x] \in I\text{)}$$

Ex. *Examples of ideals of a Lie algebra L:*

1. 0 *and L are ideals of L.*
2. *The* **derived algebra** $[LL]$, *consisting of all linear combinations of commutators $[xy]$ for all $x, y \in L$, is an ideal of L. Moreover, L is abelian iff $[LL] = 0$.*
3. *The* **center** $Z(L) := \{z \in L \mid [xz] = 0$ *for all* $x \in L\}$ *is an ideal of L. (exercise)*
4. *In $\mathfrak{gl}(n, \mathrm{F})$, $\mathfrak{n}(n, \mathrm{F}) = \{$strictly upper triangular matrices$\}$ is an ideal of $\mathfrak{t}(n, \mathrm{F}) = \{$upper triangular matrices$\}$*

Lem 1.8. *The set of inner derivations, $\operatorname{ad} L$, is an ideal of $Der(L)$. Indeed, for $z \in L$ and $\delta \in Der(L)$, we have (exercise)*

$$[\delta, \operatorname{ad} z] = \operatorname{ad} \delta(z).$$

Def. *When $I \trianglelefteq L$, the* **quotient algebra** *L/I is a Lie algebra under the operation*

$$[x + I, y + I] := [x, y] + I.$$

Def. *A Lie algebra is* **simple** *if its only ideals are 0 and L, and $[LL] \neq 0$.*

Ex. *The most famous simple algebra is $L = \mathfrak{sl}(2, \mathrm{F})$ for char $\mathrm{F} \neq 2$. $\mathfrak{sl}(2, \mathrm{F})$ has a basis:*

$$x := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad y := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

with the multiplication table

$$[xy] = h, \qquad [hx] = 2x, \qquad [hy] = -2y.$$

Check that $\mathfrak{sl}(2, \mathrm{F})$ is simple. (exercise)

Def. *The* **centralizer** *of a subset X of L is*

$$C_L(X) := \{x \in L \mid [xX] = 0\}.$$

Then $C_L(X) \leq L$, and $C_L(L) = Z(L)$.

Def. *The* **normalizer** *of a subalgebra K of L is*

$$N_L(K) := \{x \in L \mid [xK] \subset K\}.$$

$N_L(K)$ is the largest subalgebra of L which includes K as an ideal (exercise). K is called **self-normalizing** *if $N_L(K) = K$. $K \trianglelefteq L$ iff $N_L(K) = L$.*

1.2.2 Homomorphisms and representations

The definitions and basic properties of homomorphisms of Lie algebras are similar to their counterparts in linear transformations.

Def. *A linear transformation $\phi : L \to L_1$ of Lie algebras $(L, [])$ and $(L_1, []_1)$ is called a* **homomorphism** *of Lie algebras, if*

$$\phi([x, y]) = [\phi(x), \phi(y)]_1 \qquad \text{for all} \quad x, y \in L.$$

We can define the **kernel** *$Ker\,\phi$, the* **image** *$Im\,\phi$, monomorphism, epimorphism, isomorphism, and automorphism. $Ker\,\phi$ is an ideal of L; $Im\,\phi$ is a subalgebra of L_1.*

Prop 1.9. 1. *If $\phi : L \to L'$ is a homomorphism of Lie algebra, then $L/Ker\,\phi \simeq Im\,\phi$. For any ideal I of L containing in $Ker\,\phi$, there exists a unique homomorphism $\psi : L/I \to L'$, such that $\phi = \psi \circ \pi$, where $\pi : L \to L/I$ is the canonical projection.*

2. *If $I \trianglelefteq L$ and $J \trianglelefteq L$, then $(I + J)/J \simeq I/(I \cap J)$.*

3. *If $I \trianglelefteq L$, $J \trianglelefteq L$, and $I \subset J$, then $J/I \trianglelefteq L/I$ and $(L/I)/(J/I) \simeq L/J$.*

Def. *A* **representation** *of a Lie algebra L is a homomorphism $\phi : L \to \mathfrak{gl}(V)$.*

Ex. *In the adjoint representation $\mathrm{ad} : L \to \mathfrak{gl}(L)$, we have $Ker(\mathrm{ad}) = Z(L)$ and $Im(\mathrm{ad}) = \mathrm{ad}\,L$. Then $L/Z(L) \simeq \mathrm{ad}\,L$. If L is simple, then $Z(L) = 0$ and $L \simeq \mathrm{ad}\,L$.*

Def. *(optional) Every representation $\phi : L \to \mathfrak{gl}(V)$ one-to-one corresponds to a* **left L-module** *V by $x \cdot v := \phi(x)v$ for all $x \in L$ and $v \in V$. (See §6.1 for the definition of L-module)*

1.2.3 Automorphisms

Def. *Let $Aut(L)$ denote the set of all automorphisms of Lie algebra L.*

$\mathrm{Aut}\,(L)$ forms a subgroup of $\mathrm{GL}(L)$.

Prop 1.10. *Suppose $\mathrm{F} = \mathbf{R}$ or $\mathbf{C}$, L is a matrix Lie algebra, and G is the matrix group generated by $\{\exp(tx) \mid t \in \mathbf{R}, x \in L\}$. Then the linear map ϕ_g, defined by*

$$\phi_g(x) = gxg^{-1} \quad \text{for} \quad x \in L,$$

is in $Aut(L)$, and $\{\phi_g \mid g \in G\}$ is a connected normal subgroup of $Aut(L)$.

Proof. (exercise) □

Now we have the sets $\mathrm{End}\,(L) = \mathfrak{gl}(L)$, $\mathrm{Der}\,(L)$, $\mathrm{GL}(L)$, $\mathrm{Aut}\,(L)$.

1. $\mathrm{Aut}\,(L) \subset \mathrm{GL}(L) \subset \mathrm{End}\,(L)$

2. $\mathrm{Der}\,(L) \subset \mathfrak{gl}(L) = End(L)$

3. When $\mathrm{F} = \mathbf{R}$ or $\mathbf{C}$, $\langle \exp \mathfrak{gl}(L) \rangle \subset \mathrm{GL}(L)$.

Thm 1.11. *Suppose* F *is a field of characteristics 0, such that* $\exp(\mathrm{F}) \subset \mathrm{F}$ *(e.g.* $\mathrm{F} = \mathbf{R}$ *or* $\mathbf{C}$*). Then for any* $\delta \in Der(L)$*, we have*

$$\exp(\delta) := \sum_{n=0}^{\infty} \frac{1}{n!}\delta^n \ \in \ Aut(L).$$

Moreover, the group generated by $\{\exp(\delta) \mid \delta \in Der(L)\}$ *is a normal subgroup of* $Aut(L)$*.*

Def. $X \in End(L)$ *is called* **nilpotent** *if* $X^k = 0$ *for some* $k \in \mathbf{N}$*.*

Obviously, the only eigenvalue of a nilpotent endomorphism X is 0. Indeed, X can be represented as a strictly upper triangular matrix w.r.t. certain basis of L.

Cor 1.12. *Suppose* $char\,\mathrm{F} = 0$*. Given* $x \in L$ *such that* $\operatorname{ad} x$ *is nilpotent, we have* $\exp(\operatorname{ad} x) \in Aut(L)$*. Indeed,*

$$(\exp x)y(\exp x)^{-1} = \exp \operatorname{ad} x \ (y) \qquad \textit{for all} \ \ y \in L.$$