1.3 Nilpotent Lie algebras

1.3.1 Lower central series and nilpotent Lie algebras

The **lower central series** (or **descending central series**) of a Lie algebra L is given by:

$$L_{(0)} = L; \qquad L_{(k)} = [L, L_{(k-1)}], \quad k \geq 1. \tag{1.1}$$

L is called **nilpotent** if $L_{(n)} = 0$ for some n.

Ex. *Any abelian Lie algebra L is nilpotent.*

Ex. *The algebra $\mathfrak{n}(n, F)$ of strictly upper triangular matrices is nilpotent (exercise).*

Prop 1.13. *Let L be a Lie algebra.*

1. *If L is nilpotent, then so are all subalgebras and homomorphic image.*
2. *L is nilpotent iff there is a descending series of ideals*

$$L = I_0 \supset I_1 \supset I_2 \supset \cdots \supset I_r = 0,$$

such that $[L, I_k] \subseteq I_{k+1}$ for $0 \leq k \leq r - 1$.

3. *L is nilpotent iff $L/Z(L)$ is.*
4. *If L is nilpotent and nonzero, then $Z(L) \neq 0$.*

Remark. *The reverse of Prop 1.13 (1) is not true. When $I \trianglelefteq L$ is a nilpotent ideal and L/I is nilpotent, L is not necessarily nilpotent.*

Ex. *The Lie algebra of upper triangular matrices, $\mathfrak{t}(n, F)$, has the nilpotent ideal $\mathfrak{n}(n, F)$, and $\mathfrak{t}(n, F)/\mathfrak{n}(n, F) \simeq \mathfrak{d}(n, F)$ is abelian and nilpotent, but $\mathfrak{t}(n, F)$ is not nilpotent (exercise).*

1.3.2 Engel's Theorem

In last section, we have defined nilpotent endomorphisms/matrices. Now consider $\operatorname{ad} x \in \mathfrak{gl}(L)$ for $x \in L$. We call $x \in L$ **ad-nilpotent** if $\operatorname{ad} x$ is a nilpotent endomorphism.

When L is a nilpotent Lie algebra with $L_{(n)} = 0$, for any $x_1, x_2, \cdots, x_n, x \in L$ we have

$$[x_1, [x_2, [\cdots [x_n, x] \cdots]] \in L_{(n)} = 0, \quad \text{or} \quad (\operatorname{ad} x_1)(\operatorname{ad} x_2) \cdots (\operatorname{ad} x_n)(x) = 0.$$

Therefore, $(\operatorname{ad} x_1)(\operatorname{ad} x_2) \cdots (\operatorname{ad} x_n) = 0$, and so $(\operatorname{ad} L)^n = 0$. In particular, every $x \in L$ is ad-nilpotent.

The converse is also true, as stated below.

Thm 1.14 (Engel)**.** *L is a nilpotent Lie algebra iff all elements of L are ad-nilpotent.*

We need some preparations to prove Engel's theorem.

Lem 1.15. *Let $x \in \mathfrak{gl}(V)$ be a nilpotent endomorphism, then $\operatorname{ad} x$ is also nilpotent.*

Proof. x is associate with two endomorphisms of $\operatorname{End} V$: the left translation $\lambda_x(y) = xy$ and the right translation $\rho_x(y) = yx$. Since x is nilpotent, we have $x^n = 0$ for some n. Then $(\lambda_x)^n(y) = x^n y = 0$ and so $(\lambda_x)^n = 0$. Similarly, $(\rho_x)^n = 0$. Moreover, λ_x and ρ_x commute, since $\lambda_x\rho_x(y) = xyx = \rho_x\lambda_x(y)$. Therefore, $\operatorname{ad} x = \lambda_x - \rho_x$ is nilpotent, since

$$\begin{aligned}(\operatorname{ad} x)^{2n} &= (\lambda_x - \rho_x)^{2n} \\ &= \sum_{k=0}^{n} \binom{2n}{k} (-1)^{2n-k} (\lambda_x)^k (\rho_x)^{2n-k} + \sum_{k=n+1}^{2n} \binom{2n}{k} (-1)^{2n-k} (\lambda_x)^k (\rho_x)^{2n-k} = 0.\end{aligned}$$

□

Thm 1.16. *Let V be a finite dimensional vector space with $V \neq 0$, and L a subalgebra of $\mathfrak{gl}(V)$. If L consists of nilpotent endomorphisms, then there exists $v \in V$ such that $L.v = 0$.*

Proof. Use induction on $\dim L$.

1. The cases $\dim L \leq 1$ are obvious.

2. Suppose $K \neq L$ is any subalgebra of L. Since K consists of nilpotent endomorphisms, by Lemma 1.15, $\operatorname{ad} K$ acts as a Lie algebra of nilpotent endomorphisms on L, hence also on L/K. Because $\dim K < \dim L$, the induction hypothesis guarantees existence of $x + K \neq K$ in L/K killed by $\operatorname{ad} K$ action. Equivalently, $x \notin K$, and $[y, x] \in K$ for all $y \in K$. Therefore, K is properly included in $N_L(K)$.

   Now take K to be a maximal proper subalgebra of L. Then $N_L(K) = L$ by the preceding paragraph, i.e. K is an ideal of L. Fix any $z \in L - K$. Then $K + Fz$ is a subalgebra of L. Therefore, $K + Fz = L$.

   By induction, $W = \{v \in V \mid K.v = 0\}$ is nonzero. Since K is an ideal, W is stable under L:

   $$y \in K,\ x \in L,\ w \in W, \quad \text{imply} \quad yx.w = xy.w - [x, y].w = 0, \quad \text{so that} \quad x.w \in W.$$

   The nilpotent endomorphism $z \in L - K$ acts invariantly on W. There is a nonzero eigenvector $v \in W$ such that $z.v = 0$. Therefore, $L.v = 0$ as desired.

□

Proof of Engel's Theorem. Suppose L consists of ad-nilpotent elements. Then $\operatorname{ad} L \subset \mathfrak{gl}(L)$ satisfies the hypothesis of Theorem 1.16. There exists $x \in L - \{0\}$ such that $[L, x] = 0$, i.e., $Z(L) \neq 0$. Now $L/Z(L)$ consists of ad-nilpotent elements and $\dim L/Z(L) < \dim L$. Using induction on $\dim L$, we find that $L/Z(L)$ is nilpotent. Then L is nilpotent by Proposition 1.13 (3). □

Two applications of Theorem 1.16 are given below.

Cor 1.17. *If a linear Lie algebra $L \subset \mathfrak{gl}(V)$ consists of nilpotent endomorphisms, then there exists a* **flag** $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$, $\dim V_i = i$, *such that $L.V_i \subset V_{i-1}$ for all i. In other words, $L \subset \mathfrak{n}(n, F)$ with respect to a suitable basis of V.*

Lem 1.18. *Let L be nilpotent, K an ideal of L. If $K \neq 0$, then $K \cap Z(L) \neq 0$.*

Proof. $\operatorname{ad} L$ acts nilpotently on K. There is $x \in K - 0$ such that $\operatorname{ad} L(x) = [L, x] = 0$. □