## 1.4 Solvable Lie algebras

### 1.4.1 Derived series and solvable Lie algebras

The **derived series** of a Lie algebra $L$ is given by:

$$L^{(0)} = L, \quad L^{(1)} = [L, L], \quad , L^{(2)} = [L^{(1)}, L^{(1)}], \cdots , L^{(k)} = [L^{(k-1)}, L^{(k-1)}], \cdots$$

A Lie algebra $L$ is called **solvable** if $L^{(n)} = 0$ for some $n$.

**Ex.** *The Lie algebra $\mathfrak{t}(n, F)$ of $n \times n$ upper triangular matrices is solvable. (exercise)*

**Ex.** *Every nilpotent Lie algebra is solvable. (Prove $L^{(k)} \subset L_{(k)}$ for all $k$ by induction on $k$. )*

We give some basic properties of solvable Lie algebras, and compare them with those of nilpotent Lie algebras.

**Prop 1.19.** *Let $L$ be a Lie algebra.*

1. *If $L$ is solvable, then so are all subalgebras and homomorphic images of $L$.*
   *(Similar for nilpotent case)*
2. *Given any ideal $I \trianglelefteq L$, $L$ is solvable iff both $I$ and $L/I$ are solvable.*
   *(For nilpotent case, the "if" part only holds for special $I$, e.g. $I = Z(L)$.)*
3. *If $I$ and $J$ are solvable ideals of $L$, then so is $I + J$.*
   *(Similar for nilpotent case. See ex 3.6 of Humphreys)*

**Remark.** *Prop 1.19 implies that $L$ has a unique maximal nilpotent solvable ideal, called the* **radical** *of $L$, denoted $Rad\, L$. $L$ is called* **semisimple** *if $Rad\, L = 0$.*

**Cor 1.20.** *Every Lie algebra $L$ is decomposed as a solvable ideal $Rad\, L$ and a semisimple homomorphic image $L/Rad\, L$:* $\quad 0 \to Rad\, L \to L \to L/Rad\, L \to 0.$

**Ex.** *A Lie algebra $L$ is called* **simple** *if $L$ has no ideals except itself and $0$, and $[L, L] \neq 0$ (i.e. $L$ is not abelian). Show that every simple Lie algebra is semisimple.*

### 1.4.2 Lie's Theorem

In Engel's theorem, we see that every nilpotent algebra $L$ ($\dim L = n$) has $\operatorname{ad} L$ isomorphic by conjugation to a subalgebra of $\mathfrak{n}(n, F)$ (of strictly upper triangular matrices). From now on, we always assume that F is algebraically closed and char $F = 0$, unless otherwise specified. Then Lie's Theorem says that a solvable subalgebra of $\mathfrak{gl}(n, F)$ is isomorphic by conjugation to a subalgebra of $\mathfrak{t}(n, F)$ (of upper triangular matrices).

**Thm 1.21.** *Let $L$ be a solvable subalgebra of $\mathfrak{gl}(V)$, $0 < \dim V < \infty$. Then $V$ contains a common eigenvector for all the endomorphisms in $L$. Explicitly, there exist $v \in V - \{0\}$ and $\lambda \in L^*$ such that*

$$x.v = \lambda(x)v \qquad \text{for all} \;\; x \in L.$$

**Remark.** *The claim is not true if $F$ is not an algebraic closure. For example, $F = \mathbb{R}$, and*

$$S = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}, \qquad L = \left\{ \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \mid X, Y, Z \in S \right\}.$$

*Then* $L \simeq \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \mid x, y, z \in \mathbb{C} \right\}$ *is solvable. However, $L$ is not similar to a subalgebra of $\mathfrak{t}(4, \mathbb{R})$. (why?)*

*Proof of Theorem 1.21.* We prove by induction on $\dim L$. The case $\dim L = 0$ is clear. We follow the steps below similar to the proof of Theorem 1.16:

1. Goal: find a codimension one ideal $K \trianglelefteq L$; then select $z \in L - K$, so that $L = K + Fz$.

   Since $L$ is solvable, we have $L \supsetneq [L, L]$. The quotient algebra $L/[L, L]$ is abelian and every subspace is an ideal. The inverse image $K$ of a codimension one subspace of $L/[L, L]$ is a codimension one ideal in $L$.

2. Since $K$ is solvable and $\dim K < \dim L$, by induction hypothesis, there is a common eigenvector $v \in V$ for $K$. In other words, there is a linear functional $\lambda : K \to F$, such that $y.v = \lambda(y)v$ for any $y \in K$. So the *common eigenspace of $K$ for $\lambda$* is nonzero:

$$W := \{w \in V \mid y.w = \lambda(y)w, \quad \text{for all } \ y \in K\}.$$

3. Assume that $W$ is $L$-invariant. Since $F$ is algebraically closed, $z \in L$ has an eigenvector $v_0 \in W$. Since $L = K + Fz$, $v_0$ is a common eigenvector for $L$. We prove the theorem.

4. Remaining goal: Prove that $W$ is $L$-invariant.

   Fix $x \in L$ and $w \in W$. For any $y \in K$ and any integer $i > 0$,

$$y(x.w) = xy.w - [x, y].w = \lambda(y)x.w - \lambda([x, y])w. \tag{1.2}$$

   To let $x.w \in W$, we have to prove that $\lambda([x, y]) = 0$.

   Let $n > 0$ be the smallest integer for which $w, x.w, \cdots, x^n.w$ is linearly dependent. Define

$$W_i = \text{span}\{w, x.w, \cdots, x^{i-1}.w\}.$$

   Then $\dim W_i = i$ for $i < n$, and $\dim W_n = \dim W_{n+1} = \cdots = n$.

   We prove that $y(x^i.w) \in \lambda(y)x^i.w + W_i$ for any $y \in K$ and any integer $i > 0$ by induction on $i$. The case $i = 1$ is done by (1.2). Now suppose $y(x^{i-1}.w) = \lambda(y)x^{i-1}.w + w_{i-1}$, where $w_{i-1} \in W_{i-1}$. Then

$$\begin{aligned} yx^i.w &= x(yx^{i-1}.w) - [x, y]x^{i-1}.w \\ &= x(\lambda(y)x^{i-1}.w + \underbrace{w_{i-1}}_{\text{in } W_{i-1}}) - \underbrace{[x, y]x^{i-1}.w}_{\text{in } W_i} \quad \text{(by induction hypothesis)} \\ &\in \lambda(y)x^i.w + W_i. \end{aligned}$$

   The claim is proved.

   Now for any $y \in K$, all of $x$, $y$, and $[x, y] \in K$ stabilize $W_n$. Relative to the basis $w, x.w, \cdots, x^{n-1}.w$ of $W_n$, $[x, y]$ is represented by an upper triangular entries with $\lambda([x, y])$ as diagonal entries. Therefore, $0 = \text{Tr}_{W_n}([x, y]) = n\lambda([x, y])$. Since $\text{char}\, F = 0$, we have $\lambda([x, y]) = 0$ as desired. □

**Cor 1.22** (Lie's Theorem)**.** *Let $L$ be a solvable subalgebra of $\mathfrak{gl}(V)$, $\dim V < \infty$. Then $L$ stabilizes some flag in $V$. In other words, the matrices of $L$ relative to a suitable basis of $V$ are upper triangular.*

**Remark.** *Any finite dimension representation of a solvable algebra $L$, $\phi : L \to \mathfrak{gl}(V)$, implies that $\phi(L)$ is solvable and thus stabilizes a flag of $V$.*

**Cor 1.23.** *Let $L$ be solvable. Then there exists a chain of ideals of $L$, $0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$, such that* $\dim L_i = i$.

*Proof.* Apply Lie's Theorem to adjoint representation $\mathrm{ad} : L \to \mathfrak{gl}(L)$. $\square$

**Cor 1.24.** *Let $L$ be solvable. Then $x \in [L, L]$ implies that $\mathrm{ad}_L x$ is nilpotent. In particular, $[L, L]$ is nilpotent.*

*Proof.* $\mathrm{ad}\, L \subset \mathfrak{gl}(L)$ is solvable. So $\mathrm{ad}\, L$ consists of upper triangular matrices relative to a suitable basis of $L$. Then $\mathrm{ad}\,[L, L] = [\mathrm{ad}\, L, \mathrm{ad}\, L]$ consists of strictly upper triagular matrices, which are nilpotent. $\square$

### 1.4.3 Jordan-Chevalley decomposition

We explore some linear algebra results in this subsection. Let $\mathcal{J}_n(\lambda)$ denote the $n \times n$ Jordan block with diagonal entries $\lambda$, and $J_n := \mathcal{J}_n(0)$ denote the $n \times n$ matrix that has 1 on the superdiagonal and 0 elsewhere.

**Ex.**

1. *$\mathcal{J}_n(\lambda) = \lambda I_n + J_n$, where $\lambda I_n$ is diagonal, $J_n$ is nilpotent, and $\lambda I_n$ and $J_n$ are commutative;*
2. *A Jordan matrix $x = \mathcal{J}_{n_1}(\lambda_1) \oplus \mathcal{J}_{n_2}(\lambda_2) \oplus \cdots$ can be decomposed as $x = x_s + x_n$, where $x_s$ is diagonal, $x_n$ is nilpotent, $x_s$ and $x_n$ are commutative;*
3. *Given the decomposition of Jordan matrix $x = x_s + x_n$ above, any similar matrix $gxg^{-1}$ of $x$ can be decomposed as*
$$gxg^{-1} = gx_sg^{-1} + gx_ng^{-1},$$
*where $gx_sg^{-1}$ is diagonalizable, $gx_ng^{-1}$ is nilpotent, and they commute to each other.*

A matrix $x \in F^{n\times n}$ (or $x \in \mathrm{End}\,(V)$) is called **semisimple** if the roots of its minimal polynomial over $F$ are all distinct. Equivalently, when $F$ is algebraically closed, $x$ is semisimple iff $x$ is diagonalizable.

**Prop 1.25.** *Let $V$ be a finite dimensional vector space over an arbitrary field $F$, and $x \in End(V)$.*

1. *There exist unique $x_s, x_n \in End(V)$ satisfying the (additive)* **Jordan-Chevalley decomposition***: $x = x_s + x_n$, where $x_s$ is semisimple, $x_n$ is nilpotent, $x_s$ and $x_n$ commute.*
2. *There exist polynomials $p(T)$ and $q(T)$ without constant term, such that $x_s = p(x)$ and $x_n = q(x)$. In particular, $x_s$ and $x_n$ commute with any endomorphism commuting with $x$.*

*Proof for complex case:* By choosing an appropriate base of $V$, we may assume that the matrix form of $x$ is in Jordan canonical form. Then the decomposition $x = x_s + x_n$ is explicitly constructed in the proceeding example; moreover, $x$, $x_s$ and $x_n$ commute. It remains to show that $x_s$ and $x_n$ are polynomials expressions of $x$.

If $x$ has only one eigenvalue $\lambda$, then $x = \mathcal{J}_{n_1}(\lambda) \oplus \mathcal{J}_{n_2}(\lambda) \oplus \cdots \oplus \mathcal{J}_{n_k}(\lambda)$ has the Jordan-Chevalley decomposition $x = \lambda + x_n$ where $x_n = x - \lambda$. Since $(x - \lambda)^n = 0$, we can express $\lambda$ as a polynomial of $x$ without constant term, say $\lambda = p(x)$, then $x_n = x - p(x)$ is also expressed as a polynomial of $x$ without constant term.

Now suppose $x$ is a direct sum of Jordan matrices $x^{(1)} \oplus \cdots \oplus x^{(t)}$ corresponding to distinct eigenvalues $\lambda_1, \cdots, \lambda_t$, each Jordan matrix $x^{(i)}$ has the minimal polynomial $(z - \lambda_i)^{m_i}$, and $x^{(i)} = x_s^{(i)} + x_n^{(i)}$ where $x_s^{(i)} = p_i(x^{(i)})$ and $x_n^{(i)} = q_i(x^{(i)})$. The minimal polynomial of $x$ is $h(z) =$

$\prod_{i=1}^{t}(z-\lambda_i)^{m_i}$. The polynomial ring $\mathbf{C}[z]$ is PID. For each $i \in [t]$, the polynomials $(z-\lambda_i)^{m_i}$ and $h(z)/(z-\lambda_i)^{m_i} = \prod_{j\neq i}(z-\lambda_j)^{m_j}$ are relatively prime. By Chinese Remainder Theorem, there exists $r_i(z) \in \mathbf{C}[z]$ with degree less than $\deg h(z)$, such that

$$r_i(z) \equiv 1 \mod (z-\lambda_i)^{m_i}, \qquad r_i(z) \equiv 0 \mod \prod_{j\neq i}(z-\lambda_j)^{m_j}.$$

It implies that $r_i(x^{(i)}) = 1$ and $r_i(x^{(j)}) = 0$ for $j \neq i$. Therefore, let

$$p(z) = \sum_{i=1}^{t} p_i(z) r_i(z), \qquad q(z) = \sum_{i=1}^{t} q_i(z) r_i(z).$$

Then $x_s = p(x)$ and $x_n = q(x)$ as desired. □

**Remark.** *The Jordan decomposition exists in any field (See textbook, or we can modify the above proof to deal with arbitrary field.) Moreover, it is relative to the abstract Jordan decomposition for elements on any semisimple Lie algebra (later section).*

**Lem 1.26.** *Let $x \in End V$, $\dim V < \infty$, $x = x_s + x_n$ its Jordan decomposition. Then $\operatorname{ad} x = \operatorname{ad} x_s + \operatorname{ad} x_n$ is the Jordan decomposition of $\operatorname{ad} x$ (in $End(End V)$ )*

*Proof.* Suppose $\dim V = n$, and $x$ is in Jordan canonical form for a given basis of $V$. Then $\{e_{ij} \mid i,j \in [n]\}$ is a basis of $\operatorname{End} V$ in which $\operatorname{ad} x_s$ is diagonal and $\operatorname{ad} x_n$ is nilpotent. □

**Remark.** *The Jordan decomposition is indeed preserved by any finite dimensional representation $\rho : End V \to End W$, that is, $\rho(x) = \rho(x_s) + \rho(x_n)$ is the Jordan decomposition of $\rho(x)$ in $End W$.*

**Lem 1.27.** *Let $\mathcal{U}$ be a finite dimensional $F$-algebra. Then $Der\mathcal{U}$ contains the semisimple and nilpotent parts (in $End\mathcal{U}$) of all its elements.*

*Proof.* Suppose $\delta \in \operatorname{Der}\mathcal{U}$ has the Jordan decomposition $\delta = \sigma + \nu$ in $\operatorname{End}\mathcal{U}$. For $a \in F$, we set

$$\mathcal{U}_a = \{x \in \mathcal{U} \mid (\delta - a\cdot 1)^k x = 0 \text{ for some } k\}.$$

Then $\mathcal{U} = \coprod_{a\in \operatorname{spec}(\delta)} \mathcal{U}_a$ where $\operatorname{spec}(\delta) = \operatorname{spec}(\sigma)$ denotes the spectrum of $\delta$ (or $\sigma$), and $\sigma$ acts on $\mathcal{U}_a$ as a scalar multiplication by $a$. For any $n \in \mathbf{Z}^+$, by induction (exercise), the derivation $\delta$ satisfies that

$$(\delta - (a+b)\cdot 1)^n (xy) = \sum_{i=0}^{n} \binom{n}{i} ((\delta - a\cdot 1)^{n-i} x) \cdot ((\delta - b\cdot 1)^i y).$$

For any $x \in \mathcal{U}_a$ and $y \in \mathcal{U}_b$, we have $xy \in \mathcal{U}_{a+b}$. Therefore, $\sigma(xy) = (a+b)xy = \sigma(x)y + x\sigma(y)$. Therefore, $\sigma$ is a derivation. So $\sigma, \nu \in \operatorname{Der}\mathcal{U}$. □

### 1.4.4 Cartan's Criterion for Solvability

Lie's Theorem says that a linear Lie algebra $L \in \mathfrak{gl}(V)$ is solvable iff it is a subalgebra of $\mathfrak{t}(n, F)$ relative to a suitable basis. Then $[L, L]$ consists of strictly upper triangular matrices, and $\operatorname{Tr}(xy) = 0$ for all $x \in L$ and $y \in [L, L]$. It turns out that the converse is also true.

**Thm 1.28** (Cartan's Criterion)**.** *Let $L$ be a subalgebra of $\mathfrak{gl}(V)$, $V$ finite dimensional. Then $L$ is solvable iff $Tr(xy) = 0$ for all $x \in [L, L]$ and $y \in L$.*

*Proof for "$\Longleftarrow$" in complex case:* We prove that if $\mathrm{Tr}\,(xy) = 0$ for any $x \in [L, L]$ and $y \in L$, then $\mathrm{ad}\, x$ is nilpotent. By Engel's Theorem, it implies that $[L, L]$ is nilpotent, so that $L$ is solvable.

Let

$$M = \{z \in \mathfrak{gl}(V) \mid [z, L] = \mathrm{ad}\, z(L) \subseteq [L, L]\}.$$

Clearly $L \subseteq M$. Let $[x, y]$ be a typical generator of $[L, L]$, and $z \in M$. By computation (exercise),

$$\mathrm{Tr}\,([x, y]z) = \mathrm{Tr}\,(x[y, z]) = 0,$$

by $[y, z] \in [L, L]$ and the hypothesis. So $\mathrm{Tr}\,(xz) = 0$ for any $x \in [L, L]$ and $z \in M$.

Fix $x \in [L, L]$. Let $x = s + n$ be the Jordan decomposition of $x$. We show that $s = 0$, so that $\mathrm{ad}\, x = \mathrm{ad}\, n$ is nilpotent. Now $\mathrm{ad}\, x = \mathrm{ad}\, s + \mathrm{ad}\, n$ is the Jordan decomposition of $\mathrm{ad}\, s$ in $\mathrm{End}\, L$, and $\mathrm{ad}\, s$ is a polynomial of $\mathrm{ad}\, x$ without constant term. Since $\mathrm{ad}\, x(L) \subseteq [L, L]$, we have $\mathrm{ad}\, s(L) \subseteq [L, L]$ so that $s \in M$.

With an appropriate basis of $V$, $s$ has the matrix form $\mathrm{diag}\,(a_1, \cdots, a_m)$. Let $\{e_{ij} \mid i, j \in [m]\}$ be the standard basis of $\mathfrak{gl}(V)$. Then $\mathrm{ad}\, s(e_{ij}) = (a_i - a_j)e_{ij}$. So $\mathrm{ad}\, s$ is diagonal w.r.t. the basis $\{e_{ij}\}$. By Lagrange interpolation, there exists a polynomial $r(T) \in \mathbf{C}[T]$ without constant term, such that $r(a_i - a_j) = \overline{a_i - a_j}$. [1] Then $\mathrm{ad}\, \overline{s} = r(\mathrm{ad}\, s)$, which implies that $\mathrm{ad}\, \overline{s}(L) \subseteq [L, L]$. Therefore, $\overline{s} \in M$, so that

$$0 = \mathrm{Tr}\,(x\overline{s}) = \mathrm{Tr}\,(s\overline{s}) = \sum_{i=1}^{m} |a_i|^2.$$

We get $s = 0$ as desired. □

**Cor 1.29.** *A Lie algebra $L$ is solvable iff $Tr\,(\mathrm{ad}\, x\, \mathrm{ad}\, y) = 0$ for all $x \in [L, L]$, $y \in L$.*

We will recall this corollary in the next section about Killing form.

[1] This part shoud be modified for the other alg closed char 0 fields $F$. See Humphrey's text.