Chapter 2

Semisimple Lie Algebras

2.1 Killing form

2.1.1 Criterion for Semisimplicity

The **Killing form** of a Lie algebra L is a bilinear form $\kappa : L \times L \to F$ defined by:

$$\kappa(x, y) = \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y) \qquad \text{for} \quad x, y \in L.$$

The Killing form is

1. <u>symmetric</u>: $\kappa(x, y) = \kappa(y, x)$, and
2. <u>associative</u>: in the sense that $\kappa([x, y], z) = \kappa(x, [y, z])$ (since $\operatorname{Tr}([\operatorname{ad} x, \operatorname{ad} y]\operatorname{ad} z) = \operatorname{Tr}(\operatorname{ad} x[\operatorname{ad} y, \operatorname{ad} z])$.)

The **radical** of Killing form (or any symmetric bilinear form of L) is $S = \{x \in L \mid \kappa(x, y) = 0 \text{ for all } y \in L\}$. Given a basis $\{x_1, \cdots, x_n\}$ of L, the dimension of radical is

$$\dim S = n - \operatorname{rank} \left[\kappa(x_i, x_j) \right]_{n \times n}.$$

We call κ **nondegererate** if $\dim S = 0$, i.e., the matrix $\left[\kappa(x_i, x_j) \right]_{n \times n}$ is nondegenerate.

3 <u>the radical S of κ is an ideal</u>: by associativity of κ, if $x \in S$ and $y, z \in L$, then

$$\kappa([x, y], z) = \kappa(x, [y, z]) = 0 \quad \Longrightarrow \quad [x, y] \in S.$$

Ex. *Compute the matrix form of the Killing form κ of $\mathfrak{sl}(2, F)$ w.r.t. the basis $\{h, e, f\}$:*

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Lem 2.1. *Suppose a Lie algebra L has the Killing form κ, and I is an ideal of L. Then:*

1. *the Killing form of I is $\kappa_I = \kappa|_{I \times I}$;*
2. *the orthogonal subspace $I^\perp$ of I w.r.t. κ is also an ideal of L:*

$$I^\perp := \{x \in L \mid \kappa(x, y) = 0 \text{ for any } y \in I\}.$$

(Note that in general $I \cap I^\perp \neq 0$.)

Proof. For $x, y \in I$, $\operatorname{ad} x$ (resp. $\operatorname{ad} y$) maps L to I. Therefore,

$$\kappa(x,y) = \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y) = \operatorname{Tr}((\operatorname{ad} x)|_I (\operatorname{ad} y)|_I) = \kappa|_I(x,y).$$

For any $x \in I^\perp$, $y \in L$, and $z \in I$,

$$\kappa([x,y],z) = \kappa(x,[y,z]) = 0.$$

Therefore, $I^\perp$ is also an ideal of L. □

Corollary 1.29 implies that: (exercise)

a Lie algebra L is solvable iff the radical of its Killing form contains $[L, L]$.

Now we develop a criterion for L to be semisimple, i.e., the maximal solvable ideal $\operatorname{Rad} L = 0$.

Thm 2.2. *A Lie algebra L is semisimple iff its Killing form is nondegenerate.*

Proof. If L is semisimple, then $\operatorname{Rad} L = 0$. Let S be the radical of κ. Then $\operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y) = 0$ for any $x \in S$ and $y \in L$ (esp. for $y \in [S, S]$). By Corollary 1.29, S is solvable. Therefore, $S = 0$.

Conversely, suppose on the contrary, $S = 0$ but $\operatorname{Rad} L \neq 0$. Then the last nonzero term I in the derived series of $\operatorname{Rad} L$ is a nonzero abelian ideal of L (exercise). For any $x \in I$ and $y \in L$, $\operatorname{ad} x \operatorname{ad} y$ sends $L \to L \to I$. So the image of $(\operatorname{ad} x \operatorname{ad} y)^2$ is in $[I, I] = 0$. Therefore, $(\operatorname{ad} x \operatorname{ad} y)^2 = 0$, which implies that $\operatorname{ad} x \operatorname{ad} y$ is nilpotent and $\kappa(x, y) = \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y) = 0$. This shows that $I \subseteq S = 0$, a contradiction. Hence $\operatorname{Rad} L = 0$. □

Remark. *The proof also shows that $S \subseteq \operatorname{Rad} L$. However, the converse need not hold.*

Next we explore some applications of the Killing form.

2.1.2 Simple Ideals of Semisimple Lie Algebra

A Lie algebra L is a **direct sum of ideals** $L_1, \cdots, L_t$ if $L = L_1 \oplus \cdots \oplus L_t$ as vector spaces. Obviously, $[L_i, L_j] = 0$ for $i \neq j$.

Thm 2.3. *Let L be semisimple with Killing form κ. Then*

1. *L is a direct sum of some simple ideals: $L = L_1 \oplus \cdots \oplus L_t$.*

2. *The Killing form of L_i is exactly $\kappa_i = \kappa|_{L_i \times L_i}$. There is an orthogonal direct sum $\kappa = \kappa_1 \oplus \cdots \oplus \kappa_t$.*

3. *Every simple ideal of L coincides with one of the L_i.*

4. *Every ideal I of L is a direct sum of some L_i's, which is semisimple. There is a direct sum of ideals $L = I \oplus I^\perp$ w.r.t. the Killing form.*

5. *Every homomorphic image of L is semisimple and ismorphic to a direct sum of some L_i's.*

6. *$L = [L, L]$.*

Proof. Let I be any ideal of L. Then $I^\perp$ and $I \cap I^\perp$ are also ideals of L. By Cartan's Criterion, $I \cap I^\perp$ is solvable. Hence $I \cap I^\perp = 0$ and $L = I \oplus I^\perp$ by dimension counting. Moreover, any ideal J of I is also an ideal of L, and hence an orthogonal direct sum component of L w.r.t. κ. Therefore, L can be decomposed into an orthogonal direct sum of indecomposable nonabelian ideals, aka. simple ideals:

$$L = L_1 \oplus \cdots \oplus L_t, \qquad L_i \perp L_j \text{ w.r.t. } \kappa \text{ for } i \neq j.$$

Claims 1 and 2 are proved.

If I is any ideal of L, then $I = [I, L] = [I, L_1] \oplus \cdots \oplus [I, L_t]$. Each $[I, L_i] \subseteq I \cap L_i$ is either 0 or L_i. It immediately implies Claims 3, 4, and 5.

Finally,

$$[L, L] = \bigoplus_i \bigoplus_j [L_i, L_j] = \bigoplus_i [L_i, L_i] = \bigoplus_i L_i = L. \quad \square$$

Remark. *The study of semisimple Lie algebras can be done by exploring the simple Lie algebras.*

2.1.3 Derivations

We have shown that $\operatorname{ad} L$ is an ideal of $\operatorname{Der} L$. When L is semisimple, it turns out that every derivation of L is inner.

Thm 2.4. *If L is semisimple, then $\operatorname{ad} L = \operatorname{Der} L$.*

Proof. $A := \operatorname{ad} L$ is an ideal of $D := \operatorname{Der} L$. So the Killing form κ_A is the restriction of κ_D to $A \times A$. Since L is semisimple, $Z(L) = 0$ and $A \simeq L/Z(L) \simeq L$. Therefore, κ_A is nondegenerate. There is a direct sum of ideals $D = A \oplus A^\perp$ (w.r.t. the Killing form κ_D). For any $\delta \in A^\perp$ and $x \in L$,

$$0 = [\delta, \operatorname{ad} x] = \operatorname{ad}(\delta x) \quad \Longrightarrow \quad \delta x = 0 \text{ for any } x \in L.$$

Therefore, $\delta = 0$, $A^\perp = 0$, and $D = A$. $\square$

Remark. *When L is semisimple, the Lie algebra of $\operatorname{Aut} L$ is $\operatorname{Der} L = \operatorname{ad} L$. If G is a (real or complex) connected Lie group whose Lie algebra L is semisimple, then the Lie algebra of $\operatorname{Aut}(G)$ is exactly $\operatorname{Der} L = \operatorname{ad} L$.*

2.1.4 Abstract Jordan Decomposition

Lemma 1.27 shows that $\operatorname{Der} L$ contains the semisimple part and the nilpotent part of all its elements. When L is semisimple, $\operatorname{Der} L = \operatorname{ad} L$. We can write every $\operatorname{ad} x \in \operatorname{ad} L$ uniquely as

$$\operatorname{ad} x = \operatorname{ad} x_s + \operatorname{ad} x_n,$$

where $x_s, x_n \in L$, $\operatorname{ad} x_s$ is semisimple, $\operatorname{ad} x_n$ is nilpotent, and $\operatorname{ad} x_s$ and $\operatorname{ad} x_n$ commute. Then $x = x_s + x_n$ and $[x_s, x_n] = 0$. This is called the **abstract Jordan decomposition** of x in L, and x_s (resp. x_n) is called the **semisimple part** (resp. **nilpotent part**) of x.

The abstract Jordan decomposition is perserved by direct sums (exercise), Lie algebra homomorphisms, and representations (to be proved in the next section).