2.2 Complete reducibility of representations

In this section, all representations are finite dimensional. We will study a semisimple Lie algebra L by means of its adjoint representation in later sections.

2.2.1 Representations and Modules

Def. *Let L be a Lie algebra. A vector space V endowed with an operation $L \times V \to V$ (denoted $(x, v) \mapsto x.v$ or xv) is call an* **L-module** *if it satisfies the following axioms ($a, b \in F,\ x, y \in L,\ v, w \in V$):*

M1. $(ax + by).v = a(x.v) + b(y.v)$,

M2. $x.(av + bw) = a(x.v) + b(x.w)$,

M3. $[x, y].v = x.y.v - y.x.v$.

Prop 2.5. *An L-module V is equivalent to an L-representation $\rho : L \to \mathfrak{gl}(V)$ by means of:*

$$x.v \longleftrightarrow \rho(x)(v) \qquad \textit{for}\ \ x \in L,\ v \in V.$$

Every L-submodule W of V corresponds to a subrepresentation $L \to \mathfrak{gl}(W)$.

Ex. *The adjoint representation $\operatorname{ad} : L \to \mathfrak{gl}(V)$ establishes L as an L-module: $x.y = \operatorname{ad} x(y) = [x, y]$ for $x \in L$ (Lie algebra), $y \in L$ (vector space). I is submodule of L iff I is an ideal of L.*

Def. *An L-module V (equiv. representation) is called* **irreducible** *if it has precisely two L-submodules: itself and 0. The L-module V is called* **completely reducible** *if V is a direct sum of irreducible L-modules, or equivalently, if each L-submodule W of V possesses a complement L-submodule W' such that $V = W \oplus W'$ (exercise).*

Ex. *Let L be a Lie algebra.*

1. *Every one dimensional L-module is irreducible, but a zero dimensional L-module is not irreducible;*
2. *L is simple iff L is an irreducible L-module, and L is semisimple iff L is completely reducible, w.r.t. the adjoint representation of L. (exercise)*

Def. *A linear map of L-modules, $\phi : V \to W$, is called a* **homomorphism of L-modules** *if $\phi(x.v) = x.\phi(v)$ for any $x \in L$ and $v \in V$. An* **isomorphism of L-modules** *associates with a pair of* **equivalent** *representations of L.*

Other than direct sums, new L-modules can be constructed from the dual, tensors, and homomorphisms (exercises: show that they each satisfies the axioms M1, M2, and M3 for L-modules):

1. Dual: Let V be an L-module. The dual space $V^* = \operatorname{Hom}_F(V, F)$ becomes an L-module (called the **dual** or **contragredient**) such that for $x \in L$, $f \in V^*$, $v \in V$,

$$(x.f)(v) = -f(x.v).$$

2. Tensor: Let V and W be L-modules. The tensor product space $V \otimes_F W$ (or simply $V \otimes W$) becomes an L-module such that for any $x \in L$, $v \in V$ and $w \in W$,

$$x.(v \otimes w) = x.v \otimes w + v \otimes x.w.$$

3. Homomorphism and Endomorphism: Let V and W be L-modules. There is a standard isomorphism of vector spaces $\Psi : V^* \otimes_F W \to \mathrm{Hom}_F(V, W)$ defined by:

$$\Psi(\delta \otimes w)(v) := \delta(v)w, \qquad \delta \in V^*,\ w \in W,\ v \in V.$$

   The isomorphism Ψ makes $\mathrm{Hom}\,(V, W)$ an L-module by:

$$(x.f)(v) = x.f(v) - f(x.v), \qquad x \in L,\ f \in \mathrm{Hom}\,(V, W),\ v \in V.$$

   In particular, when $W = V$, then $\mathrm{End}\,(V) \simeq V^* \otimes V$ also becomes an L-module. (exercise)

The space of all bilinear forms on L is also an L-module (see Exercise 6.8 of Humphrey.)

2.2.2 Casimir element of a representation

Let L be semisimple, $\phi : L \to \mathfrak{gl}(V)$ a **faithful** (i.e. 1-1) representation of L. Then the bilinear form

$$\beta(x, y) := \mathrm{Tr}\,(\phi(x)\phi(y))$$

is symmetric, associative, and nondegenerate (similar to the Killing form where $\phi = \mathrm{ad}$).

Fix a basis $(x_1, \cdots, x_n)$ of L; there is a unique dual basis $(y_1, \cdots, y_n)$ relative to β, such that $\beta(x_i, y_j) = \delta_{ij}$. Define the **Casimir element of** ϕ by:

$$c_\phi := \sum_{i=1}^{n} \phi(x_i)\phi(y_i) \quad \in \mathrm{End}\,V.$$

Lem 2.6. *Suppose L is semisimple and $\phi : L \to \mathfrak{gl}(V)$ is a faithful representation. Then the Casimir element c_ϕ commutes with every endomorphism in $\phi(L)$.*

Proof. Given $x \in L$, let $[x, x_j] = \sum_k a_{jk} x_k$ and $[x, y_i] = \sum_j b_{ij} y_j$. The symmetry and associativity of β imply that

$$a_{ji} + b_{ij} = \beta([x, x_j], y_i) + \beta([x, y_i], x_j) = \beta(y_i, [x, x_j]) - \beta([y_i, x], x_j) = 0 \qquad \forall i, j.$$

We show that c_ϕ commutes with every endomorphism in $\phi(L)$, using the identity $[X, YZ] = [X, Y]Z + Y[X, Z]$ for $X, Y, Z \in \mathrm{End}\,V$. For $x \in L$,

$$\begin{aligned} [\phi(x), c_\phi] &= \sum_i [\phi(x), \phi(x_i)]\phi(y_i) + \sum_i \phi(x_i)[\phi(x), \phi(y_i)] \\ &= \sum_{i,j} a_{ij}\phi(x_j)\phi(y_i) + \sum_{i,j} b_{ij}\phi(x_i)\phi(y_j) \\ &= \sum_{i,j} (a_{ji} + b_{ij})\phi(x_i)\phi(y_j) = 0. \end{aligned}$$

Therefore, c_ϕ commutes with $\phi(L)$. □

When ϕ is an irreducible representation, we have $c_\phi = \dim L / \dim V$ (a scalar in $\mathrm{End}\,V$) due to the following result and the fact that $\mathrm{Tr}\,c_\phi = \sum_{i=1}^{n} \mathrm{Tr}\,(\phi(x_i)\phi(y_i)) = n = \dim L$:

Thm 2.7 (Schur's Lemma)**.** *Let $\phi : L \to \mathfrak{gl}(V)$ be irreducible. Then the only endomorphism of V commuting with all $\phi(x)$ $(x \in L)$ are the scalars.*

Proof. Suppose $y \in \mathfrak{gl}(V)$ commutes with matrices in $\phi(L)$. Let λ be an eigenvalue of y and V_λ the eigenspace of y relative to λ. For any $x \in L$ and $v \in V_\lambda$,

$$y(\phi(x).v) = \phi(x)(y.v) = \lambda\phi(x).v \quad \Longrightarrow \quad \phi(x).v \in V_\lambda.$$

In other words, V_λ is an L-submodule of V. Since $V_\lambda \neq 0$ and V is irreducible, we have $V_\lambda = V$. Therefore, $y = \lambda$ is a scalar matrix. □

2.2.3 Weyl's Theorem

Lem 2.8. *Let $\phi : L \to \mathfrak{gl}(V)$ be a finite dimensional representation of a semisimple Lie algebra L. Then $\phi(L) \subset \mathfrak{sl}(V)$. In particular, L acts trivially on any one dimensional L-module.*

Proof. Since L is semisimple, we have $\phi(L) = \phi([L, L]) \subset [\phi(L), \phi(L)] \subset \mathfrak{sl}(V)$. □

Thm 2.9 (Weyl)**.** *Every finite dimensional representation $\phi : L \to \mathfrak{gl}(V)$ of a semisimple Lie algebra L is completely reducible, that is, ϕ is a direct sum of irreducible representations of L.*

Proof. (exercise)It suffices to show that every L-submodule W of V processes a complementary L-submodule X such that $V = W \oplus X$.

First we deal with the case where W is codimension one. There are two subcases:

1. W is irreducible: Assume that L acts faithfully on V (otherwise, we replace L by $L/\mathrm{Ker}\,\phi$). Then $L.W \subset W$, and L acts trivially on the one-dimensional L-module V/W. The Casimir element $c := c_\phi$ is a linear combination of products of elements $\phi(x)$. Therefore, $c(W) \subset W$ and c acts trivially on V/W. Moreover, c commutes with $\phi(L)$. For any $x \in L$ and $v \in \mathrm{Ker}\, c$,

$$c(x.v) = c\phi(x)(v) = \phi(x)c(v) = 0 \quad \Rightarrow \quad x.v \in \mathrm{Ker}\, c.$$

   Hence $\mathrm{Ker}\, c$ is an L-submodule of V. The action of c on the irreducible L-module W is a nonzero scalar. Therefore, $\mathrm{Ker}\, c$ has dimension one, and we have $V = W \oplus \mathrm{Ker}\, c$ as desired.

2. W is not irreducible: We apply induction on $\dim W$. Let $W' \subset W$ be a nonzero L-submodule. Then W/W' is a condimension one submodule of V/W'. By induction hypothesis, $V/W' = W/W' \oplus \widetilde{W}/W'$ where $\widetilde{W}/W'$ is a module with dimension one. Now W' has codimension one in $\widetilde{W}$. By induction hypothesis, $\widetilde{W} = W' \oplus X$ where X is a submodule of dimension one. Then $V = W \oplus X$ since $W \cap X = 0$.

Next, we consider a general L-submodule W. The space $\mathrm{Hom}\,(V, W)$ is an L-module. Denote

$$\begin{aligned} \mathcal{V} &= \{f \in \mathrm{Hom}\,(V, W) \mid f|_W = a \text{ for some } a \in F\}, \\ \mathcal{W} &= \{f \in \mathrm{Hom}\,(V, W) \mid f|_W = 0\} \quad \subset \mathcal{V}. \end{aligned}$$

Both $\mathcal{V}$ and $\mathcal{W}$ are subspaces of $\mathrm{Hom}\,(V, W)$, and $\mathcal{W}$ has codimension one in $\mathcal{V}$. For any $x \in L$, $f \in \mathcal{V}$, and $w \in W$,

$$(x.f)(w) = x.f(w) - f(x.w) = a(x.w) - a(x.w) = 0.$$

So $L.\mathcal{V} \subset \mathcal{W}$, and both $\mathcal{V}$ and $\mathcal{W}$ are L-submodules of $\mathrm{Hom}\,(V, W)$. By proceeding discussion, $\mathcal{V}$ has a one-dimensional submodule (say, Fg where $g : V \to W$ has $g|_W = 1_W$) complementary to $\mathcal{W}$. Then L acts trivially on Fg, so that $0 = (x.g)(v) = x.g(v) - g(x.v)$ for $v \in V$. Therefore, $g : V \to W$ is an L-module homomorphism, and $\mathrm{Ker}\, g$ is an L-submodule of V. Since g maps V into W (so that $\dim \mathrm{Ker}\, g \geq \dim(V/W)$), and g acts as 1_W on W (so that $W \cap \mathrm{Ker}\, g = 0$), we have $V = W \oplus \mathrm{Ker}\, g$. □

Ex. *We know that every semisimple L is a direct sum of simple ideals: $L = L_1 \oplus L_2 \oplus \cdots \oplus L_t$. Then the adjoint representation* ad $: L \to \mathfrak{gl}(L)$ *is decomposed into a direct sum of irreducible representations: $L = L_1 \oplus L_2 \oplus \cdots \oplus L_t$, since each simple ideal is irreducible w.r.t. the adjoint action.*

2.2.4 Preservation of Jordan decomposition

A direct application of Weyl's Theorem lies in the Jordan decomposition.

Thm 2.10. *Let $L \subset \mathfrak{gl}(V)$ be semisimple, V finite dimensional. Then L contains the semisimple and nilpotent parts in $\mathfrak{gl}(V)$ of all its elements, and the abstract and usual Jordan decompositions in L coincide.*

Proof. Let $x \in L$ be arbitrary, with Jordan decomposition $x = x_s + x_n$ in $\mathfrak{gl}(V)$. We show that $x_s, x_n \in L$, which implies that the abstract and usual Jordan decompositions in L coincide (why?). The inclusion homomorphism $L \hookrightarrow \mathfrak{gl}(V)$ makes V an L-module. For any L-submodule W of V, define

$$L_W := \{y \in \mathfrak{gl}(V) \mid y(W) \subset W \text{ and } \mathrm{Tr}\,(y|_W) = 0\}.$$

Then $x \in L_W$ and therefore $x_s, x_n \in L_W$ since x_s and x_n are polynomials of x without constant terms. Let

$$N := N_{\mathfrak{gl}(V)}(L) = \{y \in \mathfrak{gl}(V), \mid [y, L] \subset L\}.$$

Similarly, $x \in N$ implies that $x_s, x_n \in N$. Let L' be the intersection of N with all spaces L_W. Then L' is an L-module via adjoint action, and $L \subset L' \subset N$. We have $x_s, x_n \in L'$.

Next we show that $L = L'$. Weyl's theorem implies that $L' = L \oplus M$ for an L-submodule M. Then $[L, M] \subset [L, N] \cap M \subset L \cap M = 0$. So every $y \in M$ commutes with L. However, Weyl's theorem implies that V is a direct sum of irreducible L-submodules, and Schur's Lemma shows that y acts as a scalar on each irreducible L-submodule W so that $y|_W = 0$ as $y \in L_W$. Therefore, $y = 0$ and thus $M = 0$. We get $x_s, x_n \in L' = L$. □

More preservation properties of the Jordan decomposition are listed below.

Prop 2.11. *Let L be a semisimple Lie algebra.*

1. *Direct sum: suppsose $L = L_1 \oplus \cdots \oplus L_t$ is a direct sum of ideals. Then the abs. Jordan decomposition of $x = \sum_{i=1}^t x^{(i)}$ ($x^{(i)} \in L_i$) in L is exactly $x = x_s + x_n$, where $x_s = \sum_{i=1}^t x_s^{(i)}$, $x_n = \sum_{i=1}^t x_n^{(i)}$, and $x^{(i)} = x_s^{(i)} + x_n^{(i)}$ is the abs. Jordan decomposition of each $x^{(i)}$ in L_i.*

2. *Homomorphism: if $\phi : L \to L'$ is a homomorphism between semisimple Lie algebras L and L', and $x = x_s + x_n$ is the abs. Jordan decomposition of x in L, then $\phi(x) = \phi(x_s) + \phi(x_n)$ is the abs. Jordan decomposition of $\phi(x)$ in $\phi(L)$ as well as in L'.*

3. *Representation: if $\phi : L \to \mathfrak{gl}(V)$ is a finite dimensional representation of L, and $x = x_s + x_n$ is the abs. Jordan decomposition of x in L, then $\phi(x) = \phi(x_s) + \phi(x_n)$ is the Jordan decomposition of $\phi(x)$ in $\mathfrak{gl}(V)$.*

Proof.

1. (exercise in the last section.)

2. $\phi(L) \simeq L/\mathrm{Ker}\,\phi$ where $\mathrm{Ker}\,\phi$ is an ideal of L. By complete reducibility of semisimple Lie algebras, we have $L \simeq \mathrm{Ker}\,\phi \oplus \phi(L)$. The abs. Jordan decomposition of $\phi(x)$ in $\phi(L)$ is then $\phi(x) = \phi(x_s) + \phi(x_n)$ from part 1.

   Now $\phi(L) \subset L'$. WLOG, we may assume that $L \subset \mathfrak{gl}(V)$ for some finite dimensional V. Then the abs. Jordan decompositions in $\phi(L)$ and L' both coincide with the Jordan decomposition in $\mathfrak{gl}(V)$. Therefore, $\phi(x) = \phi(x_s) + \phi(x_n)$ is the abs. Jordan decomposition of $\phi(x)$ in L'.

3. It follows from part 2 and the proceeding theorem. □

2.2.5 Reductive Lie Algebras

A Lie algebra L for which $\operatorname{Rad} L = Z(L)$ is called **reductive**. (Examples: L abelian, L semisimple, $L = \mathfrak{gl}(n, F)$.)

Thm 2.12. *The following are equivalent for a Lie algebra L:*

1. *L is reductive.*
2. *$L = Z(L) \oplus [L, L] = Z(L) \oplus L_1 \oplus \cdots \oplus L_t$ where $[L, L]$ is semisimple, and $L_1, \cdots, L_t$ are simple ideals of L.*
3. *L is a completely reducible* $\operatorname{ad} L$*-module.*

Proof.

$1 \Rightarrow 3$: If L is reductive, then $\operatorname{ad} L \simeq L/Z(L) = L/\operatorname{Rad} L$ is semisimple. Weyl Theorem implies that L is a completely reducible $\operatorname{ad} L$-module induced by the inclusion homomorphism $\operatorname{ad} L \hookrightarrow \mathfrak{gl}(L)$.

$3 \Rightarrow 2$: Suppose L is a completely reducible $\operatorname{ad} L$-module, say $L = \oplus_{i=1}^{k} L_i$ where each L_i is a simple $\operatorname{ad} L$-submodule. For any L_i, if $\dim L_i = 1$, then $\operatorname{ad} L$ acts trivially on L_i, so that $L_i \subset Z(L)$; if $\dim L_i > 1$, then L_i is a simple ideal of L and $\operatorname{ad} L$ acts faithfully on L_i, so that $L_i = \operatorname{ad} L(L_i) \subset [L, L]$. Overall, $Z(L)$ is the direct sum of those L_i with $\dim L_i = 1$, $[L, L]$ is the direct sum of those L_i with $\dim L_i > 1$, and $[L, L] \simeq L/Z(L)$ is semisimple.

$2 \Rightarrow 1$: It is obvious. □

For a reductive Lie algebra $L = Z(L) \oplus [L, L]$, its Killing form has radical $S = Z(L)$, and

$$\kappa_L = \kappa_{Z(L)} \oplus \kappa_{[L,L]} = 0|_{Z(L)} \oplus \kappa_{[L,L]}.$$

The derivation algebra of a reductive L is (exercise)

$$\operatorname{Der} L = \operatorname{End}(Z(L)) \oplus \operatorname{ad} L.$$