## 2.3 Representation of $\mathfrak{sl}(2, F)$

The representations of $\mathfrak{sl}(2, F)$ play an important role in the study of semisimple Lie algebras. In this section, we consider the finite dimensional representations of $L := \mathfrak{sl}(2, F)$, whose standard basis consists of

$$h := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad x := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad y := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

such that

$$[h, x] = 2x, \qquad [h, y] = -2y, \qquad [x, y] = h.$$

Let $V$ be an arbitrary $L$-module. Since $h$ is semisimple in $L$, the preservation of Jordan decomposition implies that $h$ acts diagonally on $V$. So $V$ is a direct sum of eigenspaces

$$V_\lambda := \{v \in V \mid h.v = \lambda v\}.$$

We let $V_\lambda = 0$ if $\lambda$ is not an eigenvalue of the $h$-action on $V$. Whenever $V_\lambda \neq 0$, we call $\lambda$ a **weight** of $h$ in $V$ and $V_\lambda$ a **weight space**.

**Lem 2.13.** *If $v \in V_\lambda$, then $x.v \in V_{\lambda+2}$ and $y.v \in V_{\lambda-2}$.*

*Proof.* $h.(x.v) = [h, x].v + x.(h.v) = 2x.v + \lambda x.v = (\lambda + 2)x.v$. Similarly for $y$. □

Since $\dim V < \infty$ and $V = \coprod_{\lambda \in F} V_\lambda$, there exists $\lambda \in F$ such that $V_\lambda \neq 0$ but $V_{\lambda+2} = 0$; this weight $\lambda$ is called a **highest weight**, and any nonzero vector $v \in V_\lambda$ is called a **maximal vector** of weight $\lambda$, where $v \neq 0$ but $x.v = 0$. We will see that a highest weight is a nonnegative integer.

Every $L$-module is a direct sum of irreducible submodules. The following theorem completely classifies all irreducible $L$-modules.

**Thm 2.14.** *Let $\phi : L \to \mathfrak{gl}(V)$ be an irreducible representation of $L = \mathfrak{sl}(2, F)$, and $\dim V = m + 1 < \infty$. Then there exists a basis $\mathcal{B} = \{v_0, v_1, \cdots, v_m\}$ of $V$, such that $\phi(h)$, $\phi(x)$ and $\phi(y)$ have the following matrix forms relative to the basis $\mathcal{B}$:*

$$\phi(h) \overset{\mathcal{B}}{\approx} \begin{bmatrix} m & & & & \\ & m-2 & & & \\ & & \ddots & & \\ & & & -(m-2) & \\ & & & & -m \end{bmatrix}, \quad \phi(x) \overset{\mathcal{B}}{\approx} \begin{bmatrix} 0 & m & & & \\ & 0 & \ddots & & \\ & & \ddots & 2 & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}, \quad \phi(y) \overset{\mathcal{B}}{\approx} \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 2 & \ddots & & \\ & & \ddots & 0 & \\ & & & m & 0 \end{bmatrix}.$$

*In particular,*

1. *For $m \in \mathbf{N}$, (up to isomorphism) there exists exactly one irreducible $L$-module of dimension $m + 1$, denoted by $V(m)$.*

2. *$V(m)$ is a direct sum of weight spaces relative to $h$: $V = \bigoplus_{i=0}^{m} V_{m-2i}$ (the irreducibility of $V(m)$ is done in homework). $V(m)$ has the highest weight $m$, and $V(m)$ has a unique (up to nonzero scalar multiplies) maximal vector in $V_m$.*

*Proof.* Let $\lambda$ be a highest weight and choose a maximal vector $v_0 \in V_\lambda - \{0\}$. Set $v_{-1} = 0$ and $v_i = (1/i!)y^i.v_0$ for $i \geq 0$. Lemma 2.13 implies that

$$h.v_i = (\lambda - 2i)v_i. \tag{2.1}$$

The definition of $v_i$ implies that

$$y.v_i = (i+1)v_{i+1}. \tag{2.2}$$

We use induction on $i$ to prove that

$$x.v_i = (\lambda - i + 1)v_{i-1} \quad \text{for} \quad i \geq 0. \tag{2.3}$$

The case $i = 0$ is obviously true. For $i > 0$,

$$\begin{aligned} ix_i = x.y.v_{i-1} = [x, y].v_{i-1} + y.x.v_{i-1} &\overset{I.H.}{=} h.v_{i-1} + (\lambda - i + 2)y.v_{i-2} \\ &= (\lambda - 2i + 2)v_{i-1} + (\lambda - i + 2)(i-1)v_{i-1} = i(\lambda - i + 1)v_{i-1}. \end{aligned}$$

Divide both sides by $i$ to complete the induction process.

The equality $h.v_i = (\lambda - 2i)v_i$ shows that $v_i \in V_{\lambda-2i}$. Hence the nonzero $v_i$ are linearly independent. By $\dim V < \infty$, there exists $n \in \mathbf{N}$ such that $v_n \neq 0$ but $v_{n+1} = 0$. Then $v_0, v_1, \cdots, v_n$ are linearly independent, and they span a nonzero $L$-submodule of $V$, which must be $V$ itself due to irreducibility of $V$. So $n = m$. By (2.3), $0 = x.v_{m+1} = (\lambda - m)v_m$, so that $\lambda = m$. Overall (2.1), (2.2), and (2.3) lead to the desired matrix forms of $\phi(h)$, $\phi(x)$, and $\phi(y)$ w.r.t. the basis $\{v_0, v_1, \cdots, v_m\}$. □

The structure of any finite dimensional $L$-module can be determined by its weight spaces as follow:

**Cor 2.15.** *Let $L = \mathfrak{sl}(2, F)$, and $V$ a finite dimensional $L$-module.*

1. *The eigenvalues of $h$ on $V$ are all integers, and each occur along its negative with equal number of times.*
2. *Suppose $V$ is decomposed into a direct sum of irreducible submodules: $V \simeq \sum_{m \in \mathbf{N}} a_m V(m)$. Then the total number of irreducible summands is*

$$\sum_{m \in \mathbf{N}} a_m = \dim V_0 + \dim V_1;$$

*for $m \in \mathbf{N}$, the number of copies of $V(m)$ in $V$ is*

$$a_m = \dim V_m - \dim V_{m+2}.$$

*Proof.* (exercise) □

In brief, given a finite dimension representation $\phi : L \to \mathfrak{gl}(V)$, the $h$-action on $V$ uniquely determines the weight spaces $V_\lambda$, their dimensions $\dim V_\lambda$, and the multiplicities $a_m$ of irreducible summands $V(m)$, in the representation $\phi$.

**Ex.** 1. *In the natural representation $\phi_1 : L \to \mathfrak{gl}(F^2)$, the standard basis $\mathcal{B}_1 := \{e_1, e_2\}$ of $F^2$ consists of eigenvectors of $\phi_1(h)$, and the matrix form $\phi_1(h) \overset{\mathcal{B}_1}{\approx} \operatorname{diag}(1, -1)$. Therefore, the $L$-module $F^2 \simeq V(1)$.*

2. *In the adjoint representation ad* $: L \to \mathfrak{gl}(L)$, *the basis* $\mathcal{B}_2 := \{x, h, y\}$ *of* $L$ *consists of eigenvectors of* $\mathrm{ad}\, h$, *and the matrix form* $\mathrm{ad}\, h \overset{\mathcal{B}_2}{\approx} \mathrm{diag}(2, 0, -2)$. *Therefore, the* $L$*-module* $L \simeq V(2)$.

3. *Consider the tensor representation* $\phi_1 \otimes \mathrm{ad} : L \to \mathfrak{gl}(F^2 \otimes L)$. *Then* $F^2 \otimes L$ *has a basis consisting of eigenvectors of* $(\phi_1 \otimes \mathrm{ad})(h)$:

$$\mathcal{B}_1 \times \mathcal{B}_2 = \{e_1 \otimes x, e_1 \otimes h, e_1 \otimes y, e_2 \otimes x, e_2 \otimes h, e_2 \otimes y\},$$

*and the matrix form of the* $h$*-action w.r.t. this basis is*

$$(\phi_1 \otimes \mathrm{ad})(h) \overset{\mathcal{B}_1 \times \mathcal{B}_2}{\approx} \mathrm{diag}(3, 1, -1, 1, -1, -3).$$

*Then*

$$(\dim V_0, \dim V_1, \dim V_2, \dim V_3, \dim V_4, \dim V_5, \cdots) = (0, 2, 0, 1, 0, 0, \cdots).$$

*We get* $a_1 = 1$, $a_3 = 1$, *and the other* $a_k = 0$. *Hence the* $L$*-module* $F^2 \otimes L = V(1) \oplus V(3)$.