2.4 Root space decomposition

Let L denote a semisimple Lie algebra in this section. We will study the detailed structure of L through its adjoint representation.

2.4.1 Maximal toral subalgebra and root space decomposition

The semisimple Lie algera L contains the semisimple part and nilpotent part of all its elements. Since L is not nilpotent, Engel's theorem ensures that there is a non-nilpotent $x \in L$, which has a nonzero semisimple part $x_s \in L$. Hence Fx_s is a nonzero subalgebra consisting of semisimple elements, called a **toral** subalgebra. So there exists a nonzero **maximal toral subalgebra** in L.

The following result is a rough analogy to Engel's theorem.

Lem 2.16. *A toral subalgebra of L is abelian.*

Proof. Suppose T is a toral subalgebra of L. We need $\operatorname{ad}_T x = 0$ for all $x \in T$. The semisimplicity of x implies that $\operatorname{ad}_T x$ is diagonalizable. If $\operatorname{ad}_T x$ has a nonzero eigenvalue α, then there is nonzero $y \in T$ such that $[x, y] = \alpha y$. Then $(\operatorname{ad}_T y)^2(x) = [y, [y, x]] = 0$. However, y is also semisimple, and $\operatorname{ad}_T y$ is diagonalizable. We can express x as a linear combination of linear independent eigenvectors $\{x_i\}$ of $\operatorname{ad}_T y$:

$$x = \sum_i a_i x_i, \qquad a_i \neq 0, \quad \operatorname{ad}_T y(x_i) = \lambda_i x_i.$$

Then $0 = (\operatorname{ad}_T y)^2(x) = \sum_i a_i \lambda_i^2 x_i$, which implies that all $\lambda_i = 0$. Therefore, $-\alpha y = (\operatorname{ad}_T y)(x) = \sum_i a_i \lambda_i x_i = 0$, which is a contradiction. So T must be abelian. □

Now fix a maximal toral subalgebra H of L. Then $\operatorname{ad}_L H$ is a commuting family of semsimple elements. So $\operatorname{ad}_L H$ is simiultaneously diagonalizable (exercise). L is the direct sum of some common eigenspaces of $\operatorname{ad}_L H$:

$$L_\alpha = \{x \in L \mid [h, x] = \alpha(h)x\}, \qquad \alpha \in H^*. \tag{2.4}$$

In particular,

$$H \subseteq L_0 = C_L(H), \qquad \text{the centralizer of } H \text{ in } L. \tag{2.5}$$

Each nonzero $\alpha \in H^*$ for which $L_\alpha \neq 0$ is called a **root**; the set of all roots is denoted by Φ. Then we get the **root space decomposition**:

$$L = C_L(H) \oplus \coprod_{\alpha \in \Phi} L_\alpha. \tag{2.6}$$

Ex. *Suppose $L = \mathfrak{sl}(\ell + 1, \mathrm{F})$. Let $h_i := e_{i,i} - e_{i+1,i+1}$ for $i \in [\ell]$. Then $H := \sum_{i \in [\ell]} \mathrm{F} h_i$ is a maximal torus subalgebra of L, and $C_L(H) = H$. Denote by $\epsilon_i \in H^*$ the linear functional that takes the i-th diagonal entry of elements of H. Then each e_{ij} ($i, j \in [\ell + 1]$, $i \neq j$) is a common eigenvector for elements of H, such that for $h \in H$, $(\operatorname{ad}_L h)(e_{ij}) = [h, e_{ij}] = (\epsilon_i - \epsilon_j)(h) e_{ij}$. Therefore, $\mathfrak{sl}(\ell + 1, \mathrm{F})$ has the root space decomposition (exercise)*

$$\begin{aligned} \mathfrak{sl}(\ell+1, \mathrm{F}) &= H \oplus \coprod_{\substack{i \neq j \\ i,j \in [\ell+1]}} \mathrm{F} e_{ij}, \qquad \textit{where} \\ \textit{the set of roots} \quad \Phi &:= \{\epsilon_i - \epsilon_j \mid i, j \in [\ell+1],\ i \neq j\}. \end{aligned}$$

Thm 2.17. *1. For $\alpha, \beta \in H^*$, $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$. In particular, if $\alpha \neq 0$ and $x \in L_\alpha$, then $\operatorname{ad} x$ is nilpotent.*

2. If $\alpha, \beta \in H^*$ and $\alpha + \beta \neq 0$, then $L_\alpha \perp L_\beta$ relative to the Killing form κ of L.

Proof. Let $\alpha, \beta \in H^*$, $h \in H$, $x \in L_\alpha$, $y \in L_\beta$ be arbitrary.

1.
$$\operatorname{ad} h([x,y]) = [[h,x],y] + [x,[h,y]] = \alpha(h)[x,y] + \beta(h)[x,y] = (\alpha+\beta)(h)[x,y].$$

So $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$. If $\alpha \neq 0$ and $x \in L_\alpha$, then $\operatorname{ad} x$ is nilpotent according to the root space decomposition of L.

2. If $\alpha + \beta \neq 0$, then

$$\begin{aligned} \kappa([h,x],y) = -\kappa([x,h],y) = -\kappa(x,[h,y]) \quad &\Rightarrow \quad \alpha(h)\kappa(x,y) = -\beta(h)\kappa(x,y) \\ &\Rightarrow \quad (\alpha+\beta)(h)\kappa(x,y) = 0 \quad \text{for any } h \in H \\ &\Rightarrow \quad \kappa(x,y) = 0. \end{aligned}$$

Hence $L_\alpha \perp L_\beta$ relative to the Killing form κ of L. □

Theorem 2.17 and the nondegeneracy of κ in L imply $\kappa(L_\alpha, L_{-\alpha}) \neq 0$ and the following result.

Cor 2.18. *The restriction of the Killing form to $L_0 = C_L(H)$ is nondegenerate.*

Proof. In the root space decomposition $L = C_L(H) \oplus \coprod_{\alpha \in \Phi} L_\alpha$, $C_L(H) = L_0$ is orthogonal to all L_α for $\alpha \in \Phi$ relative to κ. Since κ is nondegenerate on L, it must be nondegenerate on $C_L(H)$ as well. □

Remark. *The restriction of κ to C is $\kappa(x,y) = Tr(\operatorname{ad} x \operatorname{ad} y)$ for $x, y \in C$, instead of $\kappa'(x,y) = Tr(\operatorname{ad}_C x \operatorname{ad}_C y)$. In fact, H is in the radical of κ', so that κ' is dengerate.*

Ex. *We verify Theorem 2.17 for $L = \mathfrak{sl}(\ell+1, \mathrm{F})$ with the root space decomposition:*

$$\mathfrak{sl}(\ell+1,\mathrm{F}) = H \oplus \coprod_{\substack{i \neq j \\ i,j \in [\ell+1]}} \mathrm{F}e_{ij}$$

1. The root space $\mathrm{F}e_{ij} = L_{\epsilon_i - \epsilon_j}$. We have

$$[L_{\epsilon_i-\epsilon_j}, L_{\epsilon_p-\epsilon_q}] = [\mathrm{F}e_{ij}, \mathrm{F}e_{pq}] = \delta_{jp}\mathrm{F}e_{iq} - \delta_{qi}\mathrm{F}e_{pj} = \left\{\begin{matrix} 0 & \text{if } p \neq j,\ q \neq i \\ \mathrm{F}e_{iq} & \text{if } p = j,\ q \neq i \\ \mathrm{F}e_{pj} & \text{if } p \neq j,\ q = i \\ \mathrm{F}(e_{ii} - e_{jj}) & \text{if } p = j,\ q = i \end{matrix}\right\} \subseteq L_{\epsilon_i-\epsilon_j+\epsilon_p-\epsilon_q}.$$

2. In $\mathfrak{sl}(\ell+1,\mathrm{F})$, it is known that $\kappa(x,y) = 2(\ell+1)\mathrm{Tr}(xy)$. We can examine the orthogonal relationship of the root spaces, using the basis

$$\mathcal{B} := \{h_i \mid i \in [\ell]\} \cup \{e_{ij} \mid i,j \in [\ell+1], i \neq j\}.$$

(a) $\kappa(h_i, h_i) = 4(\ell+1)$, $\kappa(h_i, h_{i+1}) = -2(\ell+1)$, $\kappa(h_i, h_j) = 0$ *for* $|j-i| > 1$. *In particular, a basis dual to* $\{h_i \mid i \in [\ell]\}$ *consists of*

$$g_i := \frac{\ell+1-i}{2(\ell+1)^2}\sum_{s=1}^{i} e_{ss} - \frac{i}{2(\ell+1)^2}\sum_{t=i+1}^{\ell+1} e_{tt}. \tag{2.7}$$

We can verify that $\kappa(h_i, g_j) = \delta_{ij}$.

(b) $\kappa(h_i, e_{pq}) = 2(\ell+1)\mathrm{Tr}(h_i e_{pq}) = 2(\ell+1)(\epsilon_p - \epsilon_q)(h_i)\mathrm{Tr}(e_{pq}) = 0$. *So* $H \perp \mathrm{F}e_{pq}$.

(c) $\kappa(e_{ij}, e_{pq}) = 2(\ell+1)\mathrm{Tr}(e_{ij}e_{pq}) = \begin{cases} 2(\ell+1), & (p,q) = (j,i), \\ 0, & \text{otherwise.} \end{cases}$

So $\mathrm{F}e_{ij} \perp \mathrm{F}e_{pq}$ *whenever* $(p,q) \neq (j,i)$.

The above case study shows that $L_\alpha \perp L_\beta$ *whenever* $\alpha + \beta \neq 0$.

Thm 2.19. *Any maximal toral subalgebra* H *of* L *satisfies that* $H = C_L(H)$.

Proof. The proof is proceeded in several steps:

(1) *C contains the semisimple and nilpotent parts of its elements.* If $x \in C$, then $\operatorname{ad} x(H) \subseteq H$, so that

$$\operatorname{ad} x_s(H) = (\operatorname{ad} x)_s(H) \subseteq H, \qquad \operatorname{ad} x_n(H) = (\operatorname{ad} x)_n(H) \subseteq H.$$

Hence $x_s, x_n \in C$.

(2) *All semisimple elements of C lie in H.* If $x \in C$ is semisimple, then $H + Fx$ is an abelian subalgebra consisting of semisimple elements. Since H is a maximal toral subalgbra, we have $H + Fx = H$ and thus $x \in H$.

(3) *C is nilpotent.* For any $x \in C$, $x = x_s + x_n$. The semisimple part x_s lies in H, so that $\operatorname{ad}_C x_s = 0$, and thus $\operatorname{ad}_C x = \operatorname{ad}_C x_n$ is nilpotent. By Engel's Theorem, C is nilpotent.

(4) *The restriction of* κ *to H is nondegenerate.* Let $\kappa(h, H) = 0$ for some $h \in H$; we show that $h = 0$. If $x \in C$ is nilpotent, then $\operatorname{ad} x$ is nilpotent and commutes with $\operatorname{ad} h$ (since $[x,h] = 0$). So $\operatorname{ad} x\ \operatorname{ad} h$ is nilpotent and $\kappa(x,h) = \mathrm{Tr}\,(\operatorname{ad} x\ \operatorname{ad} h) = 0$. By (1) and (2), we see that $\kappa(h, C) = 0$, which forces $h = 0$ by the nondegeneracy of κ on C.

(5) *C is abelian.* Otherwise $[C,C] \neq 0$, C being nilpotent by (3). Then $Z(C) \cap [C,C] \neq 0$, since the nilpotent $\operatorname{ad} C$-action on $[C,C]$ annihilates a nonzero element z, which must be in $Z(C) \cap [C,C]$. The associativity of κ implies that $\kappa(H, [C,C]) = 0$. The nondegeneracy of κ on H by (4) implies that $H \cap [C,C] = 0$. So $z \notin H$, and its nilpotent part $z_n \in C - \{0\}$. Then $\operatorname{ad} z_n$ is nilpotent and commutes with $\operatorname{ad} C$, so that $\kappa(z_n, C) = 0$, contarary to the nondegeneracy of κ on C (Corollary 2.18).

(6) $C = H$. Otherwise C contains a nonzero nilpotent element x by (1), (2). Then $\operatorname{ad} x$ is nilpotent, and $[x, C] = 0$ by (5) implies that $\kappa(x, C) = \mathrm{Tr}\,(\operatorname{ad} x\ \operatorname{ad} C) = 0$, contrary to the nondegeneracy of κ on C. □

Now the root space decomposition of L becomes

$$L = H \oplus \coprod_{\alpha \in \Phi} L_\alpha \tag{2.8}$$

where $L_0 = H = C_L(H)$; every $h \in H$ is semisimple; for $\alpha \in \Phi$, every $x \in L_\alpha$ is nilpotent; relative to the Killing form κ, $L_\alpha \perp L_\beta$ for any $\alpha + \beta \neq 0$.

Cor 2.20. *The restriction of κ to $H = C_L(H)$ is nondegenerate.*

The corollary implies that there is a bijection $H^* \to H$ such that every $\alpha \in H^*$ corresponds to the unique $t_\alpha \in H$ that satisfies

$$\alpha(h) = \kappa(t_\alpha, h) \qquad \text{for all } \ h \in H. \tag{2.9}$$

Ex. *The Lie algebra $L = \mathfrak{sl}(\ell+1, \mathrm{F})$ has the root system $\Phi = \{\epsilon_i - \epsilon_j \mid i, j \in [\ell+1],\ i \neq j\}$. Let us compute t_α for $\alpha = \epsilon_i - \epsilon_j$. Recall that we have $\kappa(x, y) = 2(\ell+1)\mathrm{Tr}(xy)$ for $L = \mathfrak{sl}(\ell+1, \mathrm{F})$. Suppose $t_\alpha = \sum_{k=1}^{\ell+1} t_k e_{kk} \in H$, then for any $h = \sum_{k=1}^{\ell+1} s_k e_{kk} \in H$, we have*

$$\kappa(t_\alpha, h) = 2(\ell+1) \sum_{k=1}^{\ell+1} t_k s_k = \alpha(h) = (\epsilon_i - \epsilon_j)(h) = s_i - s_j.$$

Therefore,

$$t_{\epsilon_i - \epsilon_j} = \frac{1}{2(\ell+1)}(e_{ii} - e_{jj}).$$

2.4.2 Properties of the Root Space Decomposition

The root space decomposition and the Killing form imply a rich structure of L.

Thm 2.21. *The Φ, H, and L_α have the following properties:*

1. *Φ spans H^*.*

2. *If $\alpha \in \Phi$, then $-\alpha \in \Phi$; both L_α and $L_{-\alpha}$ have dimension 1; for $x \in L_\alpha$, $y \in L_{-\alpha}$, we have*

$$[x, y] = \kappa(x, y) t_\alpha. \tag{2.10}$$

In particular, $[L_\alpha, L_{-\alpha}] = \mathrm{F} t_\alpha$. The t_α satisfies that $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$. For any nonzero $x_\alpha \in L_\alpha$, there exists unique $y_\alpha \in L_{-\alpha}$ such that $S_\alpha := \mathrm{span}\,(x_\alpha, y_\alpha, h_\alpha := [x_\alpha, y_\alpha])$ is a simple subalgebra of L isomorphic to $\mathfrak{sl}(2, \mathrm{F})$ via

$$x_\alpha \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad y_\alpha \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad h_\alpha \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.11}$$

Moreover, $h_\alpha = \dfrac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)} = -h_{-\alpha}$ is independent of the choice of x_α.

3. *If $\alpha \in \Phi$, then $\mathrm{F}\alpha \cap \Phi = \{\pm\alpha\}$.*

4. *If $\alpha, \beta \in \Phi$, then $\beta(h_\alpha) \in \mathbf{Z}$, and $\beta - \beta(h_\alpha)\alpha \in \Phi$. (The numbers $\beta(h_\alpha)$ are called* **Cartan integers***.)*

5. *Let $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$. Let r, q be (resp.) the largest integers for which $\beta - r\alpha$, $\beta + q\alpha$ are roots. Then all $\beta + i\alpha \in \Phi$ for $-r \leq i \leq q$, and the Cartan integer $\beta(h_\alpha) = r - q$.*

6. *If $\alpha, \beta, \alpha + \beta \in \Phi$, then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.*

Proof. 1. If Φ does not span H^*, there exists a nonzero $h \in H$ s.t. $\alpha(h) = 0\ \forall \alpha \in \Phi$. So $[h, L_\alpha] = 0\ \forall \alpha \in \Phi$. Moreover, $[h, H] = 0$. Hence $h \in Z(\mathrm{L}) = 0$, a contradiction.

2. If $\alpha \in \Phi$ but $-\alpha \notin \Phi$, then $\alpha + \beta \neq 0$ for any $\beta = 0$ or $\beta \in \Phi$, so that $\kappa(L_\alpha, L_\beta) = 0$. Thus $\kappa(L_\alpha, L) = 0$, contradicting the nondegeneracy of κ. So $-\alpha \in \Phi$.

   Suppose $x \in L_\alpha$, $y \in L_{-\alpha}$ are nonzero. Then for any $h \in H$,

$$\begin{aligned} \kappa(h, [x,y] - \kappa(x,y)t_\alpha) &= \kappa(h, [x,y]) - \kappa(h, \kappa(x,y)t_\alpha) \\ &= \kappa([h,x], y) - \kappa(x,y)\kappa(h, t_\alpha) = \alpha(h)\kappa(x,y) - \alpha(h)\kappa(x,y) = 0. \end{aligned}$$

   Therefore, $[x,y] = \kappa(x,y)t_\alpha$. The relation also holds if one of x and y is zero. Now $\kappa(L_\alpha, L_{-\alpha}) \neq 0$ since otherwise $\kappa(L_\alpha, L) = 0$, a contradiction to nondegeneracy of κ. So $[L_\alpha, L_{-\alpha}] = \mathrm{F}t_\alpha$.

   If $0 = \kappa(t_\alpha, t_\alpha) = \alpha(t_\alpha)$, then $[t_\alpha, x] = 0 = [t_\alpha, y]$. The Lie subalgebra $S = \mathrm{span}(x, y, t_\alpha)$ is solvable. So $\mathrm{ad}_L S$ is solvable, and Lie's theorem implies that $\mathrm{ad}_L t_\alpha \in \mathrm{ad}_L[S,S]$ is nilpotent. Then t_α is nilpotent. However, $t_\alpha \in H$ is semisimple. We have $t_\alpha = 0$, a contradiction. Hence $\kappa(t_\alpha, t_\alpha) \neq 0$.

   For any nonzero $x_\alpha \in L_\alpha$, let $y_\alpha \in L_{-\alpha}$ such that $\kappa(x_\alpha, y_\alpha) = \dfrac{2}{\kappa(t_\alpha, t_\alpha)}$; let $h_\alpha := [x_\alpha, y_\alpha]$. Clearly

$$h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)} = -h_{-\alpha}, \quad [h_\alpha, x_\alpha] = \alpha(h_\alpha)x_\alpha = 2x_\alpha, \quad [h_\alpha, y_\alpha] = -2y_\alpha.$$

   So the Lie subalgebra $S_\alpha := \mathrm{span}(x_\alpha, y_\alpha, h_\alpha)$ is isomorphic to $\mathfrak{sl}(2, \mathrm{F})$ via (2.11).

   Consider the subspace M of L spanned by H along with all root spaces of the form $L_{c\alpha}$ ($c \in \mathrm{F}^*$). M is a S_α-submodule so that it is a direct sum of irreducible S_α-modules $V(m)$ for $m \in \mathbf{Z}^+$. The $\mathrm{Ker}\,\alpha$ is a codimension 1 subspace in H complement to $\mathrm{F}h_\alpha$, that is, $\mathrm{Ker}\,\alpha \oplus \mathrm{F}h_\alpha = H$. Since S_α acts trivially on $\mathrm{Ker}\,\alpha$, the $V(m)$ components of M with even integers m is $S_\alpha \oplus \mathrm{Ker}\,\alpha$. In particular, $2\alpha \notin \Phi$. Then $\frac{1}{2}\alpha \notin \Phi$ as well (otherwise, replacing α by $\frac{1}{2}\alpha$ and we will get contradiction). So M doesn't have $V(m)$ summands for odd integers m. Hence

$$M = S_\alpha \oplus \mathrm{Ker}\,\alpha. \tag{2.12}$$

   In particular, $\dim L_\alpha = \dim L_{-\alpha} = 1$.

3. It is obvious by (2.12).

4. For $\beta \in \Phi$, we consider the S_α-action on L_β. The subspace

$$M_\beta := \sum_{i \in \mathbf{Z}} L_{\beta + i\alpha} \tag{2.13}$$

   is a S_α-module of L. In particular, L_β is a weight space of h_α with the weight $\beta(h_\alpha)$, which must be in $\mathbf{Z}$. Then $-\beta(h_\alpha) \in \mathbf{Z}$ is also a weight of M_β. Since $-\beta(h_\alpha) = (\beta - \beta(h_\alpha)\alpha)(h_\alpha)$, we have $\beta - \beta(h_\alpha)\alpha \in \Phi$.

5. Now assume $\beta \neq \pm\alpha$ in the above paragraph. Then no $\beta + i\alpha$ can be 0. Each root space is one dimensional, and the integral weights appearing in M_β has the form $(\beta + i\alpha)(h_\alpha) = \beta(h_\alpha) + 2i$, which covers 0 or 1 exactly once. Therefore, M_β is an irreducible S_α-module. Let r, q be (resp.) the largest integers for which $\beta - r\alpha$, $\beta + q\alpha$ are roots. Then all $\beta + i\alpha \in \Phi$ for $-r \leq i \leq q$. The highest (resp. lowest) weight is $(\beta + q\alpha)(h_\alpha) = \beta(h_\alpha) + 2q$ (resp. $\beta(h_\alpha) - 2r$.) The symmetry of weights implies that $\beta(h_\alpha) + 2q + \beta(h_\alpha) - 2r = 0$, that is, $\beta(h_\alpha) = r - q$. We call $\beta - r\alpha, \cdots, \beta, \cdots, \beta + q\alpha$ the **α-string through** β.

6. In the S_α-module M_β, obviously $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.

□

Consider F = **C** case. Let

$$E := \sum_{\alpha\in\Phi} \mathbf{R}\alpha$$

be the real span of Φ. Then $E \subseteq H^*$ with $\dim_{\mathbf{R}} E = \dim_{\mathrm{F}} H^*$ since Φ spans H^*. We may tranfer the nondegenerate Killing form from H to H^* by

$$(\gamma, \delta) := \kappa(t_\gamma, t_\delta) \qquad \text{for all} \;\; \gamma, \delta \in H^*. \tag{2.14}$$

Some properties of Φ can be rephrased as follow.

Thm 2.22. *Let (Φ, E) be defined from (L, H) as above. Then:*

1. *Φ spans E, and $0 \notin \Phi$.*
2. *If $\alpha \in \Phi$, then $-\alpha \in \Phi$, and no other scalar multiple of α is in Φ.*
3. *If $\alpha, \beta \in \Phi$, then $\beta - \dfrac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \Phi$, where $\dfrac{2(\beta,\alpha)}{(\alpha,\alpha)} = \beta(h_\alpha) \in \mathbf{Z}$ is a Cartan integer.*

The set Φ is called a **root system** in E. We will completely classify all (Φ, E) for simple Lie algebras in the next Chapter.