Chapter 3

Root Systems

3.1 Introduction

3.1.1 Reflections in a euclidean space

A **eucludean space** E is a finite dimensional vector space endowed with an inner product $(,)$ (i.e. a positive definite symmetric bilinear form). The main objective of this chapter is the root systems in a fixed euclidean space E.

Every vector $\alpha \in E$ has an orthogonal complement in E: $P_\alpha := \alpha^\perp = \{\beta \in E \mid (\alpha, \beta) = 0\}$, which has codimension 1 and is called a **hyperplane**. There is a unique $\sigma_\alpha \in \mathrm{End}(E)$ s.t.

$$\sigma_\alpha(\alpha) = -\alpha, \qquad \sigma_\alpha|_{P_\alpha} = id_{P_\alpha}. \tag{3.1}$$

In other words, $\sigma_\alpha \in \mathrm{End}(E)$ satisfies that $\sigma_\alpha^2 = 1$, σ_α has P_α as $+1$-eigenspace and $\mathbf{R}\alpha$ as -1-eigenspace. σ_α is called a **reflection**. Obviously, σ_α preserves the lengths of vectors in E, so that σ_α preserves $(,)$.

Every $\beta \in E$ can be decomposed as

$$\beta = \left(\beta - \frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha\right) + \frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha, \qquad \left(\beta - \frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha\right) \perp \alpha.$$

Then the explicit form of σ_α is given by:

$$\begin{aligned} \sigma_\alpha(\beta) &= \sigma_\alpha\left(\beta - \frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha\right) + \sigma_\alpha\left(\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha\right) = \left(\beta - \frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha\right) - \frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha \\ &= \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha. \end{aligned} \tag{3.2}$$

We define

$$\langle \beta, \alpha \rangle := \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \tag{3.3}$$

which is linear in the first component. So $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$.

Lem 3.1. *Suppose Φ is a finite set that spans E, and all reflections σ_α ($\alpha \in \Phi$) leave Φ invariant. If $\sigma \in \mathrm{GL}(E)$ leaves Φ invariant, sends a nonzero $\alpha \in \Phi$ to $-\alpha$, and fixes pointwise a hyperplane P of E, then $\sigma = \sigma_\alpha$.*

Proof. Let $\tau = \sigma\sigma_\alpha$.

1. τ acts as identity on $\mathbf{R}\alpha$ as well as on $E/\mathbf{R}\alpha$. So all eigenvalues of τ are 1.
2. $\tau(\Phi) = \Phi$, so that τ permutes the elements of Φ. There exists $k \in \mathbf{Z}^+$ such that τ^k fixes all elements of Φ. Then $\tau^k = 1$ since Φ spans E. So τ is diagonalizable.

Overall, $\tau = 1$ and $\sigma = \sigma_\alpha^{-1} = \sigma_\alpha$. □

3.1.2 Root systems

Def. *A* **root system** *of euclidean space E is a subset Φ of E that statisfies:*

(R1) Φ is finite, spans E, and does not contain 0.

(R2) If $\alpha \in \Phi$, then $\mathbf{R}\alpha \cap \Phi = \{\pm\alpha\}$.

(R3) If $\alpha \in \Phi$, the reflection σ_α leaves Φ invariant.

(R4) If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbf{Z}$.

Def. *The group $\mathcal{W}$ generated by reflections $\{\sigma_\alpha \mid \alpha \in \Phi\}$ is a finite subgroup of permutations of Φ. We call $\mathcal{W}$ the* **Weyl group** *of Φ.*

Ex. *For semisimple Lie algebras L, the Φ we obtained in the last section are root systems.*

Lem 3.2. *Let Φ be a root system in E, with Weyl group $\mathcal{W}$. If $\sigma \in \mathrm{GL}(E)$ leaves Φ invariant, then $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$ for all $\alpha \in \Phi$, and $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$ for all $\alpha, \beta \in \Phi$.*

Proof. First, σ permutes Φ; so does σ^{-1}. Hence $\sigma\sigma_\alpha\sigma^{-1}$ leaves Φ invariant. Second, $\sigma\sigma_\alpha\sigma^{-1}$ fixes the hyperplane $\sigma(P_\alpha)$, and

$$\sigma\sigma_\alpha\sigma^{-1}(\sigma(\alpha)) = \sigma\sigma_\alpha(\alpha) = -\sigma(\alpha).$$

Therefore, Lemma 3.1 implies that $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$. Moreover,

$$\begin{aligned} \sigma_{\sigma(\alpha)}(\sigma(\beta)) &= \sigma(\beta) - \langle \sigma(\beta), \sigma(\alpha) \rangle \sigma(\alpha), \\ \sigma\sigma_\alpha\sigma^{-1}(\sigma(\beta)) &= \sigma\left(\beta - \langle \beta, \alpha \rangle \alpha\right) = \sigma(\beta) - \langle \beta, \alpha \rangle \sigma(\alpha). \end{aligned}$$

Thus $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$. □

For a root system Φ, any pair $\alpha, \beta \in \Phi$ should make $\langle \alpha, \beta \rangle, \langle \beta, \alpha \rangle \in \mathbf{Z}$. Let $\theta \in [0, \pi]$ be the angle between α and β. Then

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)} \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = 4\cos^2\theta \in \mathbf{Z}.$$

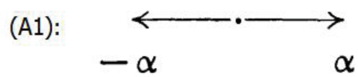
The possible values of $4\cos^2\theta$ are: $0, 1, 2, 3, 4$. Therefore, when $\beta \neq \pm\alpha$, the angle θ could be $\pi/2, \pi/3, 2\pi/3, \pi/4, 3\pi/4, \pi/6, 5\pi/6$. Table 1 (p.45) lists all possible combinations in a root system.

Table 1.

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ	$\Vert\beta\Vert^2 / \Vert\alpha\Vert^2$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
−1	−1	$2\pi/3$	1
1	2	$\pi/4$	2
−1	−2	$3\pi/4$	2
1	3	$\pi/6$	3
−1	−3	$5\pi/6$	3

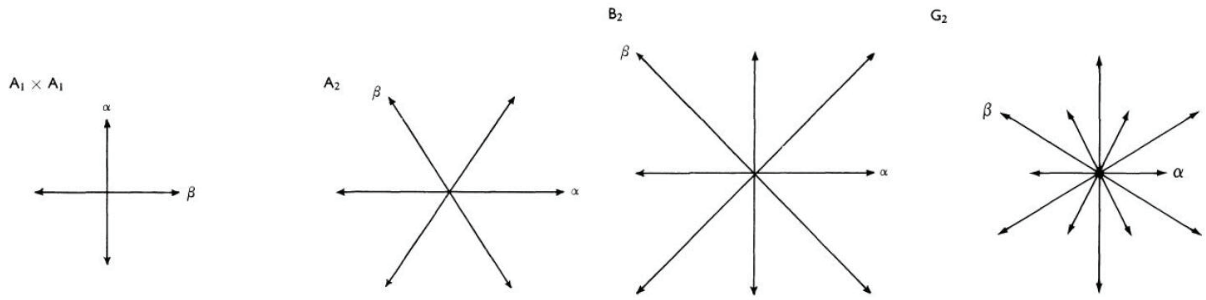
Def. *We call $\ell := \dim E$ the* **rank** *of the root system Φ.*

Ex. *The only rank 1 root system is:*



We can associate it with the root system of $\mathfrak{sl}(2, \mathrm{F})$.

Ex. *The rank 2 root systems include 4 cases:* *(see p.44)*



The first two could be associate with the root systems of $\mathfrak{sl}(2, \mathrm{F}) \times \mathfrak{sl}(2, \mathrm{F})$ and $\mathfrak{sl}(3, \mathrm{F})$ respectively.

Table 1 also implies the following lemma:

Lem 3.3. *Suppose $\alpha, \beta \in \Phi$ and $\beta \neq \pm\alpha$. If $(\alpha, \beta) > 0$, then $\alpha - \beta$ is a root. If $(\alpha, \beta) < 0$, then $\alpha + \beta$ is a root.*

Proof. If $(\alpha, \beta) > 0$, then $\langle \alpha, \beta \rangle > 0$. Table 1 shows that at least one of $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$ equals 1. If $\langle \alpha, \beta \rangle = 1$, then $\sigma_\beta(\alpha) = \alpha - \beta \in \Phi$. Similarly, if $\langle \beta, \alpha \rangle = 1$, then $\sigma_\alpha(\beta) = \beta - \alpha \in \Phi$, so that $\alpha - \beta \in \Phi$. The proof is similar for $(\alpha, \beta) < 0$. □

As an application, if $\alpha, \beta \in \Phi$ and $\beta \neq \pm\alpha$, and $r, q \in \mathbf{N}$ are the largest integers such that $\beta - r\alpha$ and $\beta + q\alpha$ are roots, then we can use the lemma to show that $\beta + i\alpha \in \Phi$ for any integer i with $-r \leq i \leq q$ (why?). By $\sigma_\alpha(\beta + q\alpha) = \beta - r\alpha$, we will get $r - q = \langle \beta, \alpha \rangle$, so that any α-string through β has length at most 4 (since we can start at an extreme vector with $r = 0$ or $q = 0$).