3.2 Simple Roots and Weyl Group

In this section, we fix a root system Φ of rank ℓ in a euclidean space E, with Weyl group $\mathcal{W}$.

3.2.1 Bases and Weyl chambers

Def. *A subset Δ of Φ is called a* **base** *if:*

(B1) Δ is a basis of E,

(B2) Every root $\beta \in \Phi$ can be written as $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$ with integral coefficients k_α all nonnegative or all nonpositive.

When a base Δ exists, clearly $|\Delta| = \ell$.

- *The roots in Δ are called* **simple roots***.*
- *The* **height** *of a root $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$ is $\mathrm{ht}\beta := \sum_{\alpha\in\Delta} k_\alpha$.*
- *Define a* **partial order** *$\succ$ in E, such that $\lambda \succ \mu$ iff $\lambda - \mu$ is a sum of positive roots or 0. Then every root $\beta \in \Phi$ has either $\beta \succ 0$ (***positive***) or $\beta \prec 0$ (***negative***) .*
- *The collection of positive roots (resp. negative roots) relative to Δ is denoted Φ^+ (resp. Φ^-). Obviously, $\Phi = \Phi^+ \sqcup \Phi^-$.*

Ex. *Find a base for each of the root systems with $\ell = 1$ or 2. Determine the heights and partial orders of the roots w.r.t. the base.*

Lem 3.4. *If Δ is a base of Φ, then for any $\alpha \neq \beta$ in Δ, $(\alpha, \beta) \leq 0$ and $\alpha - \beta$ is not a root.*

Proof. By (B2), $\alpha - \beta$ cannot be a root. Therefore $(\alpha, \beta) \leq 0$ by Lemma 3.3. □

We will proves the existence and constructs all possible bases of Φ.

Def.

1. *A vector $\gamma \in E$ is called* **regular** *if $\gamma \in E - \bigcup_{\alpha\in\Phi} P_\alpha$, that is, no $\alpha \in \Phi$ such that $(\gamma, \alpha) = 0$; otherwise, γ is called* **singular***.*
2. *For $\gamma \in E$, define*
$$\Phi^+(\gamma) := \{\alpha \in \Phi \mid (\gamma, \alpha) > 0\},$$
which consists of the roots lying on the positive side of P_γ.
3. *Call $\alpha \in \Phi^+(\gamma)$* **decomposable** *if $\alpha = \beta_1 + \beta_2$ for some $\beta_i \in \Phi^+(\gamma)$,* **indecomposable** *otherwise.*

Thm 3.5. *Let $\gamma \in E$ be regular. Then the set $\Delta(\gamma)$ of all indecomposable roots in $\Phi^+(\gamma)$ is a base of Φ, and every base is obtainable in this manner.*

Proof. It is proceeded in steps.

1. Each root in $\Phi^+(\gamma)$ is a nonnegative $\mathbf{Z}$-linear combination of $\Delta(\gamma)$. Otherwise some $\alpha \in \Phi^+(\gamma)$ cannot be so written; choose α that minimizes (γ, α). Obviously, $\alpha \notin \Delta(\gamma)$, so $\alpha = \beta_1 + \beta_2$ for some $\beta_i \in \Phi^+(\gamma)$, whence $(\gamma, \alpha) = (\gamma, \beta_1) + (\gamma, \beta_2)$. The regularity of γ implies that $(\gamma, \beta_i) < (\gamma, \alpha)$, so that β_1 and β_2 must be $\mathbf{Z}$-linear combinations of $\Delta(\gamma)$, whence α also is, which is a contradiction.

2. If $\alpha, \beta \in \Delta(\gamma)$ and $\alpha \neq \beta$, then $(\alpha, \beta) \leq 0$. Otherwise $\alpha - \beta \in \Phi$ by Lemma 3.3. If $\alpha - \beta \in \Phi^+$, then $\alpha = (\alpha - \beta) + \beta$ is decomposable, contradicting $\alpha \in \Delta(\gamma)$; otherwise $\beta - \alpha \in \Phi^+$, which implies the contradiction that $\beta = (\beta - \alpha) + \alpha$ is decomposable.

3. $\Delta(\gamma)$ is a linearly independent set. Otherwise $\sum r_\alpha \alpha = 0$ for $\alpha \in \Delta(\gamma)$, $r_\alpha \in \mathbf{R}$ and some $r_\alpha \neq 0$. Seperating the positive coefficientss from the negative ones, we may rewrite it as $\sum s_\alpha \alpha = \sum t_\beta \beta$, where $s_\alpha, t_\beta > 0$, and the sets of α's and β's are disjoint. Let $\epsilon := \sum s_\alpha \alpha$. Then
$$0 \leq (\epsilon, \epsilon) = \sum_{\alpha, \beta} s_\alpha t_\beta (\alpha, \beta) \leq 0.$$
So that $\epsilon = 0$ and $0 = (\gamma, \epsilon) = \sum s_\alpha (\gamma, \alpha)$. The regularity of γ forces all $s_\alpha = 0$. Similarly, all $t_\beta = 0$. (*The argument shows that any set of vectors lying strictly on one side of a hyperplane in E and forming pairwise obtuse angles must be linearly independent.*)

4. $\Delta(\gamma)$ is a base of Φ. The regularity of γ implies that $\Phi = \Phi^+(\gamma) \cup -\Phi^+(\gamma)$. By (1), $\Delta(\gamma)$ satisfies (B2) and spans $\Phi^+(\gamma)$, whence it spans Φ and E.

5. Each base Δ of Φ has the form $\Delta(\gamma)$ for some regular $\gamma \in E$. Given Δ, select $\gamma \in E$ so that $(\gamma, \alpha) > 0$ for all $\alpha \in \Delta$. Then γ is regular and $\Phi^+ \subseteq \Phi^+(\gamma)$, $\Phi^- \subseteq -\Phi^+(\gamma)$. So $\Phi^+ = \Phi^+(\gamma)$, and Δ must consist of indecomposable elements. Hence $\Delta \subseteq \Delta(\gamma)$, which forces $\Delta = \Delta(\gamma)$ since both are bases of E. □

The hyperplanes P_α ($\alpha \in \Phi$) partition E into finitely many regions; the connected components of $E - \bigcup_{\alpha \in \Phi} P_\alpha$ are called the (open) **Weyl chambers** of E. Each regular $\gamma \in E$ belongs to exactly one Weyl chamber, denoted $\mathcal{C}(\gamma)$.

Lem 3.6. *Weyl chambers are in 1-1 correspondence with bases, such that $\mathcal{C}(\gamma) \leftrightarrow \Delta(\gamma)$ for any regular $\gamma \in E$.*

Proof. Two chambers $\mathcal{C}(\gamma) = \mathcal{C}(\gamma')$ iff γ and γ' lie on the same side of each hyperplane P_α ($\alpha \in \Phi$), iff $\Phi^+(\gamma) = \Phi^+(\gamma')$, iff $\Delta(\gamma) = \Delta(\gamma')$. □

Write $\mathcal{C}(\Delta) = \mathcal{C}(\gamma)$ if $\Delta = \Delta(\gamma)$, and call it the **fundamental Weyl chamber relative to Δ.** So $\mathcal{C}(\Delta)$ is the open convex set intersected by all positive sides of P_α ($\alpha \in \Delta$), with P_α ($\alpha \in \Delta$) as boundary wall; a vector $\beta \in \mathcal{C}(\Delta)$ iff $(\beta, \alpha) > 0$ for all $\alpha \in \Delta$.

The reflections in the Weyl group $\mathcal{W}$ can send a Weyl chamber onto any neighborhood chamber, whence $\mathcal{W}$ acts transitively onto Weyl chambers. It implies that $\mathcal{W}$ acts transitively on the bases: for any $\sigma \in \mathcal{W}$, $\gamma \in E$ regular, $\alpha \in E$,
$$(\sigma\gamma, \sigma\alpha) = (\gamma, \alpha), \qquad \sigma(\Delta(\gamma)) = \Delta(\sigma(\gamma)), \qquad \sigma(\mathcal{C}(\gamma)) = \mathcal{C}(\sigma(\gamma)).$$

3.2.2 Further properties of simple roots

Let Δ be a base of a root system Φ in E. We list some useful auxiliary results on simple roots.

Lem 3.7. *If α is a non-simple positive root, then there is $\beta \in \Delta$ such that $\alpha - \beta$ is a positive root.*

Proof. If $(\alpha, \beta) \leq 0$ for all $\beta \in \Delta$, then $\Delta \cup \{\alpha\}$ is a set of vectors lying strictly on one side (i.e. $\Phi^+(\gamma)$) of a hyperplane and be pairwise obtuse. Therefore, $\Delta \cup \{\alpha\}$ is a linearly independent set, which is absurd. Hence $(\alpha, \beta) > 0$ for some $\beta \in \Delta$, so that $\alpha - \beta \in \Phi$. Clearly $\alpha - \beta \in \Phi^+$. □

Cor 3.8. *Each positive root $\beta \in \Phi^+$ can be written as $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ ($\alpha_i \in \Delta$, not necessarily distinct) such that each partial sum $\alpha_1 + \cdots + \alpha_i$ is a positive root.*

Proof. Use the lemma and induction on $\mathrm{ht}\beta$. □

Lem 3.9. *For any simple root $\alpha \in \Delta$, the reflection σ_α permutes the positive roots other than α.*

Proof. Let $\beta \in \Phi^+ - \{\alpha\}$ be expressed as $\beta = \sum_{\gamma \in \Delta} k_\gamma \gamma$ ($k_\gamma \in \mathbf{N}$). Then $k_\mu > 0$ for some $\mu \in \Delta - \{\alpha\}$. The root $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$ has a term $k_\mu \mu$ with positive coefficient k_μ. Therefore, $\sigma_\alpha(\beta) \in \Phi^+$; moreover, $\sigma_\alpha(\beta) \neq \alpha$ since $\sigma_\alpha(-\alpha) = \alpha$. □

Cor 3.10. *Let $\delta = \dfrac{1}{2} \sum_{\beta \succ 0} \beta$. Then $\sigma_\alpha(\beta) = \delta - \alpha$ for all $\alpha \in \Delta$.*

Lem 3.11. *Let $\sigma_1, \cdots, \sigma_t \in \Delta$ (not necessarily distinct). Write $\sigma_i = \sigma_{\alpha_i}$. If $\sigma_1 \cdots \sigma_{t-1}(\alpha_t) \prec 0$, then there is an index $1 \leq s < t$ such that*

$$\sigma_1 \cdots \sigma_t = \sigma_1 \cdots \sigma_{s-1} \sigma_{s+1} \cdots \sigma_{t-1}.$$

Proof. Write $\beta_i = \sigma_{i+1} \cdots \sigma_{t-1}(\alpha_t)$, $0 \leq i \leq t-2$, $\beta_{t-1} = \alpha_t$. Then $\beta_0 \prec 0$ and $\beta_{t-1} \succ 0$. We can find the least index s such that $\beta_s \succ 0$. Then $\sigma_s(\beta_s) = \beta_{s-1} \prec 0$, and Lemma 3.9 implies $\beta_s = \alpha_s$. Recall that $\sigma_{\sigma(\alpha)} = \sigma \sigma_\alpha \sigma^{-1}$ for $\sigma \in \mathcal{W}$ and $\alpha \in \Phi$ (Lemma 3.2). Therefore,

$$\sigma_s = \sigma_{\alpha_s} = \sigma_{\beta_s} = \sigma_{\sigma_{s+1} \cdots \sigma_{t-1}(\alpha_t)} = (\sigma_{s+1} \cdots \sigma_{t-1}) \sigma_t (\sigma_{t-1} \cdots \sigma_{s+1}),$$

which leads to $\sigma_1 \cdots \sigma_t = \sigma_1 \cdots \sigma_{s-1} \sigma_{s+1} \cdots \sigma_{t-1}$. □

Cor 3.12. *If $\sigma \in \mathcal{W}$ is expressed as $\sigma = \sigma_1 \cdots \sigma_t$, where σ_i are reflections corresponding to simple roots and t is minimal, then $\sigma(\alpha_t) \prec 0$.*

3.2.3 The Weyl Group

We have made the one-to-one correspondence between bases and Weyl chambers of Φ. Now we show that the Weyl group $\mathcal{W}$ is generated by all "simple relfections", and $\mathcal{W}$ acts simply transitively on Weyl chambers and bases.

Thm 3.13. *Let Δ be a base of Φ.*

1. *If $\gamma \in E$ is regular, then there is $\sigma \in \mathcal{W}$ such that $(\sigma(\gamma), \alpha) > 0$ for all $\alpha \in \Delta$. So $\mathcal{W}$ acts transitively on Weyl chambers.*

2. *If Δ' is another base of Φ, then $\sigma(\Delta') = \Delta$ for some $\sigma \in \mathcal{W}$. So $\mathcal{W}$ acts transitively on bases.*

3. *If α is any root, then there is $\sigma \in \mathcal{W}$ such that $\sigma(\alpha) \in \Delta$.*

4. *$\mathcal{W}$ is generated by σ_α ($\alpha \in \Delta$).*

5. *If $\sigma \in \Delta$ satsifies that $\sigma(\Delta) = \Delta$, then $\sigma = 1$. So $\mathcal{W}$ acts simply transitively on bases.*

Proof. Let $\mathcal{W}'$ denote the subgroup generated by all σ_α ($\alpha \in \Delta$). We prove the first three statements for $\mathcal{W}'$ first; then show that $\mathcal{W}' = \mathcal{W}$.

1. Let $\delta = \frac{1}{2}\sum_{\beta \succ 0} \beta$ and choose $\sigma \in \mathcal{W}'$ for which $(\sigma(\gamma), \delta)$ is maximal. For every $\alpha \in \Delta$, we have $\sigma_\alpha \sigma \in \mathcal{W}'$ so that
$$(\sigma(\gamma), \delta) \geq (\sigma_\alpha\sigma(\gamma), \delta) = (\sigma(\gamma), \sigma_\alpha(\delta)) = (\sigma(\gamma), \delta - \alpha) = (\sigma(\gamma), \delta) - (\sigma(\gamma), \alpha).$$
Hence $(\sigma(\gamma), \alpha) = (\gamma, \sigma^{-1}(\alpha)) > 0$ since γ is regular.

2. $\mathcal{W}'$ permutes the Weyl chambers, hence it also permutes the bases.

3. It suffices to show that each root belongs to at least one base. For any root $\alpha \in \Phi$, we can choose γ closed enough to P_α such that $(\gamma, \alpha) = \epsilon > 0$, but $|(\gamma, \beta)| > \epsilon$ for all $\beta \in \Phi - \{\pm\alpha\}$. Then $\alpha \in \Delta(\gamma)$ by the construction of base $\Delta(\gamma)$.

4. For each $\alpha \in \Phi$, there exists $\sigma \in \mathcal{W}'$ such that $\beta = \sigma(\alpha) \in \Delta$. Then $\sigma_\alpha = \sigma_{\sigma^{-1}(\beta)} = \sigma^{-1}\sigma_\beta\sigma \in \mathcal{W}'$. Therefore, $\mathcal{W}'$ contains all reflections σ_α ($\alpha \in \Phi$) and hence $\mathcal{W}' = \mathcal{W}$.

5. If $\sigma \in \Delta$ satsifies that $\sigma(\Delta) = \Delta$ but $\sigma \neq 1$, then σ can be expressed as a product of simple reflection(s) with minimal length, say $\sigma = \sigma_1 \cdots \sigma_t$ where $\sigma_i := \sigma_{\alpha_i}$, $\alpha_i \in \Delta$. Then Corollary 3.12 shows that $\sigma(\alpha_t) \prec 0$, which is a contradiction.

□

Some additiona properties of Weyl group actions are listed below (See [Humphrey] for the proofs). When $\sigma \in \mathcal{W}$ is expressed as $\sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$ ($\alpha_i \in \Delta$, t minimal), we call it a **reduced** expression, and call $\ell(\sigma) := t$ the **length** of σ relative to Δ; we define $\ell(1) := 0$. Define
$n(\sigma) =$ number of positive roots α for which $\sigma(\alpha) \prec 0$.

Lem 3.14. *For all $\sigma \in \mathcal{W}$, $\ell(\sigma) = n(\sigma)$.*

The next lemma shows that each vector in E is $\mathcal{W}$-conjugate to precisely one point of the closure of fundamental Weyl chamber $\overline{\mathcal{C}(\Delta)}$.

Lem 3.15. *Let $\lambda, \mu \in \overline{\mathcal{C}(\Delta)}$. If $\sigma\lambda = \mu$ for some $\sigma \in \mathcal{W}$, then σ is a product of simple reflections which fix λ. In particular $\lambda = \mu$.*

Ex. *Discuss the simply transitive $\mathcal{W}$-actions on Weyl chambers and bases for root systems A_2.*

3.2.4 Irreducible Root Systems

A root system Φ (resp. a base Δ) is called **irreducible** if it cannot be partitioned into the union of two proper orthogonal subsets, i.e. each root in one subset is orthogonal to each root in the other. Φ is irreducible iff any base Δ is. Every root system can be expressed as a product of irreducible root systems.

Lem 3.16. *Let Φ be irreducible. Then $\mathcal{W}$ acts irreducibly on E. In particular, the $\mathcal{W}$-orbit of a root α spans E.*

Proof. Suppose $E' \subseteq E$ is nonzero $\mathcal{W}$-invariant subspace of E. Let $E'' := (E')^{\perp}$. Then $E = E' \oplus E''$. For any $\alpha \in \Phi$, the reflection σ_α has $\dim 1$ eigenspace $\mathbf{C}\alpha$ and $\dim(\ell - 1)$ eigenspace P_α. On the other hand, E' is σ_α-invariant, so that E' is spanned by some eigenvectors of σ_α. If $\alpha \notin E'$, then all eigenvectors of σ_α correspond to -1 eigenvalue, whence $E' \subseteq P_\alpha$ and $\alpha \in E''$. This shows that every root in Φ is either in E' or in E''. However, Φ is irreducible. Therefore $E'' = 0$ and $E' = E$. □

Lem 3.17. *Let Φ be irreducible. Then at most two root lengths occur in Φ, and all roots of a given length are conjugate under $\mathcal{W}$.*

Proof. Let $\alpha, \beta \in \Phi$ be arbitrary roots. Since $\mathcal{W}(\beta)$ spans E (Lemma 3.16), there is $\sigma \in \mathcal{W}$ such that $\sigma(\beta)$ is not orthogonal to α. If $(\alpha, \beta) \neq 0$, the possible ratios of squared root lengths of α, β are $1, 2, 3, 1/2, 1/3$. Hence it is impossible to have three different root lengthes in Φ.

Now suppose α and β have equal length. If $\alpha \notin \mathcal{W}(\beta)$, we may assume $(\alpha, \beta) \neq 0$ (otherwise, replace β by $\sigma(\beta)$ for some $\sigma \in \mathcal{W}$.) Then $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = \pm 1$. Replacing β (if needed) by $-\beta = \sigma_\beta(\beta)$, we may asusme that $\langle \alpha, \beta \rangle = 1$. Then

$$(\sigma_\alpha \sigma_\beta \sigma_\alpha)(\beta) = (\sigma_\alpha \sigma_\beta)(\beta - \alpha) = \sigma_\alpha(-\beta - \alpha + \beta) = \alpha,$$

which contradicts $\alpha \notin \mathcal{W}(\beta)$. Therefore $\alpha \in \mathcal{W}(\beta)$. □

When Φ is irreducible with two distinct root lengths, we call them **long roots** and **short roots** respectively.

When Φ is irreducible, each closed Weyl chamber contains exactly one root of each root length. (exercise)

Lem 3.18. *Let Φ be irreducible. Then there is a unique maximal root β relative to $\prec$. In particular, for any $\alpha \in \Phi$, $\alpha \neq \beta$ implies $ht\,\alpha < ht\,\beta$, and $(\beta, \alpha) \geq 0$ for all $\alpha \in \Delta$; if $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$, then all $k_\alpha > 0$. Moreover, if Φ has two distinct root lengths, then β is a long root.*

Proof. Let $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ be a maximal root relative to $\prec$. Then $\beta \succ 0$, so that each $k_\alpha \geq 0$. Denote

$$\Delta_1 = \{\alpha \in \Delta \mid k_\alpha > 0\}, \qquad \Delta_2 = \{\alpha \in \Delta \mid k_\alpha = 0\}.$$

Then $\Delta = \Delta_1 \sqcup \Delta_2$. If $\Delta_2 \neq \emptyset$, then by the irreducibility of Φ and Δ, we can find $\alpha_1 \in \Delta_1$ and $\alpha_2 \in \Delta_2$ such that $(\alpha_1, \alpha_2) < 0$. Therefore, $(\beta, \alpha_2) < 0$, whence $\beta + \alpha_2 \in \Phi$, a contradiction to the maximality of β. So all $k_\alpha > 0$.

The above argument shows that $(\beta, \alpha) \geq 0$ for all $\alpha \in \Delta$. Moreover, at least one $(\beta, \alpha) > 0$ since β is a positive linear combination of all $\alpha \in \Delta$.

Now if β' is another maximal root, then $(\beta, \beta') > 0$ so that $\beta - \beta'$ is a root unless $\beta = \beta'$. However, $\beta - \beta'$ cannot be a root since otherwise $\beta \prec \beta'$ or $\beta' \prec \beta$. Therefore, $\beta = \beta'$ is unique.

Finally, if Φ has two root lengths, then for any $\alpha \in \overline{\mathcal{C}(\Delta)}$, we have $\beta - \alpha \succ 0$, so that $(\gamma, \beta - \alpha) \geq 0$ for any $\gamma \in \overline{\mathcal{C}(\Delta)}$. Let $\gamma := \beta$, then $(\beta, \beta) \geq (\beta, \alpha)$; let $\gamma := \alpha$, then $(\beta, \alpha) \geq (\alpha, \alpha)$. Therefore, $(\beta, \beta) \geq (\alpha, \alpha)$. Since every root is $\mathcal{W}$-conjugate to a root in $\overline{\mathcal{C}(\Delta)}$ with the same root length, β must be a long root. □