3.3 Classification of Simple Lie Algebras

3.3.1 Cartan matrices, Coxeter graphs, and Dynkin diagrams

Fix an ordering $(\alpha_1, \cdots, \alpha_\ell)$ of the simple roots. The matrix $\left[\langle \alpha_i, \alpha_j \rangle\right]$ is called the **Cartan matrix** of Φ, and the entries $\langle \alpha_i, \alpha_j \rangle$ are called **Cartan integers**.

Thm 3.19. *The Cartan matix of Φ determines Φ up to isomorphism. Precisely, if $\Phi' \subseteq E'$ is another root system with a base $\Delta' = \{\alpha'_1, \cdots, \alpha'_\ell\}$, such that $\langle \alpha_i, \alpha_j \rangle = \langle \alpha'_i, \alpha'_j \rangle$ for $1 \leq i, j \leq \ell$, then the bijection $\alpha_i \mapsto \alpha'_i$ extends to an isomorphism $\phi : E \to E'$ mapping Φ to Φ' and $\langle \phi(\alpha), \phi(\beta) \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \Phi$.*

Proof. For $\alpha, \beta \in \Delta$,

$$\sigma_{\phi(\alpha)}(\phi(\beta)) = \phi(\beta) - \langle \phi(\beta), \phi(\alpha) \rangle \phi(\alpha) = \phi(\beta) - \langle \beta, \alpha \rangle \phi(\alpha) = \phi(\beta - \langle \beta, \alpha \rangle \alpha) = \phi(\sigma_\alpha(\beta)).$$

So $\sigma_{\phi(\alpha)} = \phi \circ \sigma_\alpha \circ \phi^{-1}$. The Weyl groups $\mathcal{W}$ and $\mathcal{W}'$ are generated by simple reflections. So $\sigma \mapsto \phi \circ \sigma \circ \phi^{-1}$ gives the isomorphism of $\mathcal{W}$ and $\mathcal{W}'$. Every root $\beta \in \Phi$ is conjugate under $\mathcal{W}$ to a simple root, say $\beta = \sigma(\alpha)$ for $\alpha \in \Delta$. Then $\phi(\beta) = (\phi \circ \sigma \circ \phi^{-1})(\phi(\alpha)) \in \Phi'$. So ϕ maps Φ onto Φ'. Moreover, the formula for a reflection shows that ϕ preserves all Cartan integers. □

Given the ordered base $(\alpha_1, \cdots, \alpha_\ell)$ of Φ, the **Coxeter graph** of Φ is a graph having ℓ vertices, with the ith joined to the jth ($i \neq j$) by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges. Clearly, $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle \in \{0, 1, 2, 3\}$, and it is greater than 1 only if α_i and α_j have different root lengths.

When a double or triple edge occurs in the Coxeter graph, we may add an arrow pointing the the shorter of the two roots. The resulting figure is called the **Dynkin diagram** of Φ.

A root system is irreducible iff its Coxeter graph (resp. Dynkin diagram) is connected.

Ex. *The Cartan matrix for rank 1 root system A_1 is $\begin{bmatrix} 2 \end{bmatrix}$. The Cartan matrices for root systems of rank 2 are:*

$$A_1 \times A_1 : \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad A_2 : \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad B_2 : \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}, \quad G_2 : \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$$

Work out the corresponding Coxeter graphs and Dynkin diagrams. (exercise)

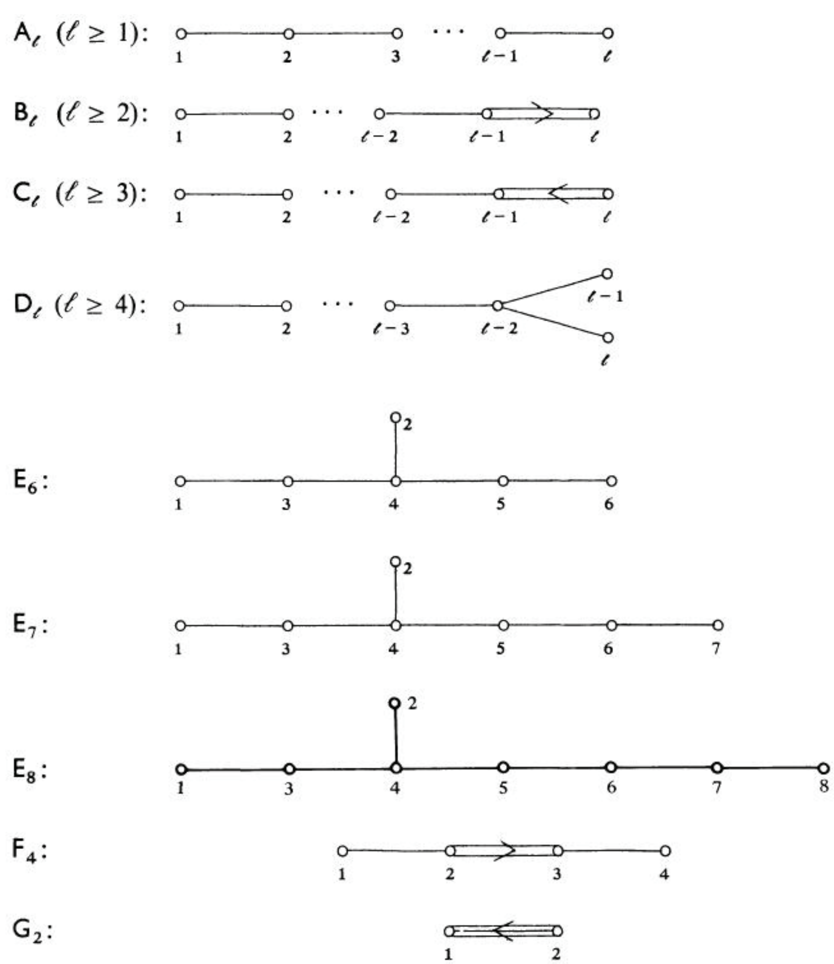
3.3.2 Classification of irreducible root systems

Let $\Phi = \Phi_1 \sqcup \Phi_2 \sqcup \cdots \sqcup \Phi_t$ be the decomposition of Φ into irreducible root systems, $\Delta = \Delta_1 \sqcup \Delta_2 \sqcup \cdots \sqcup \Delta_t$ the corresponding partition of Δ, and E_i the span of Δ_i. Then E has the orthogonal direct sum $E = E_1 \oplus E_2 \oplus \cdots \oplus E_t$; each Φ_i is an irreducible root system in E_i with base Δ_i and Weyl group $\mathcal{W}_i$ generated by all σ_α ($\alpha \in \Delta_i$). The Coxetor graph (resp. Dynkin diagram) of Φ is the disjoint union of those of Φ_i. Moreover, $\mathcal{W} = \mathcal{W}_1 \times \mathcal{W}_2 \times \cdots \times \mathcal{W}_t$; so each E_i is $\mathcal{W}$-invariant.

Thm 3.20. *Φ decomposes uniquely as the union of irreducible root systems Φ_i in subspaces E_i of E such that $E = E_1 \oplus E_2 \oplus \cdots \oplus E_t$ (orthogonal direct sum).*

It remains to classify all irreducible root systems.

Thm 3.21. *If Φ is an irreducible root system of rank ℓ, then its Dynkin diagram is one of the following (ℓ vertices in each case):*



The corresponding Cartan matrices is in Table 1:

Table 1. *Cartan matrices*

$$A_\ell: \begin{pmatrix} 2 & -1 & 0 & & & \cdot & \cdot & \cdot & & 0 \\ -1 & 2 & -1 & 0 & & \cdot & \cdot & \cdot & & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & & \cdot & \cdot & \cdot & -1 & 2 \end{pmatrix}$$

$$B_\ell: \begin{pmatrix} 2 & -1 & 0 & & \cdot & \cdot & \cdot & & & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \cdot & \cdot & \cdot & -1 & 2 & -2 \\ 0 & 0 & 0 & & \cdot & \cdot & \cdot & 0 & -1 & 2 \end{pmatrix}$$

$$C_\ell: \begin{pmatrix} 2 & -1 & 0 & & \cdot & \cdot & \cdot & & & 0 \\ -1 & 2 & -1 & & \cdot & \cdot & \cdot & & & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot & \cdot & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \cdot & \cdot & \cdot & -1 & 2 & -1 \\ 0 & 0 & & & \cdot & \cdot & \cdot & 0 & -2 & 2 \end{pmatrix}$$

$$D_\ell: \begin{pmatrix} 2 & -1 & 0 & & & \cdot & \cdot & \cdot & & & 0 \\ -1 & 2 & -1 & & & \cdot & \cdot & \cdot & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & & \cdot & \cdot & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & & \cdot & \cdot & & -1 & 2 & -1 & -1 \\ 0 & 0 & & \cdot & \cdot & & 0 & -1 & 2 & 0 \\ 0 & 0 & & \cdot & \cdot & & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$E_6: \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$E_7: \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$E_8: \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$F_4: \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \qquad G_2: \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Proof. If $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$ is an irreducible base, then the angle θ_{ij} between α_i and α_j has $\cos\theta_{ij} \leq$

0 and $4\cos^2\theta_{ij} = \langle\alpha_i, \alpha_j\rangle\langle\alpha_j, \alpha_i\rangle \in \{0, 1, 2, 3\}$. Define an **admissible** set to be a set of linearly independent <u>unit</u> vectors $\mathcal{U} = \{e_1, \cdots, e_n\}$ such that

$$\cos\delta_{ij} = (e_i, e_j) \leq 0 \quad \text{and} \quad 4(e_i, e_j)^2 \in \{0, 1, 2, 3\} \quad \text{for any } i \neq j.$$

Here δ_{ij} denotes the angle between e_i and e_j. We attach a graph Γ to $\mathcal{U}$ such that e_i and e_j are joint by $4(e_i, e_j)^2$ edges. Obviously, every Coxeter graph is an admissible set $\mathcal{U}$ with the graph Γ.

1. If some of the e_i are discarded, the remaining ones still form an admissible set.

2. The number of pairs of vertices in Γ connected by at least one edge is strictly less than n. Let $e = \sum_i e_i$, which is nonzero. Then

$$0 < (e, e) = n + \sum_{i<j} 2(e_i, e_j).$$

   The pair (e_i, e_j) $(i < j)$ is connected by at least one edge iff $4(e_i, e_j)^2 = 1, 2, 3$, iff $2(e_i, e_j) \leq -1$. The number of such pairs cannot exceed $n - 1$.

3. Γ contains no cycles. Otherwise, a cycle Γ' corresponds to an admissible set $\mathcal{U}'$ by (1), which violates (2) by replacing n with $\mathrm{Card}\,\mathcal{U}'$.

4. No more than three edges can originate at a given vertex of Γ. Suppose $e \in \mathcal{U}$ has edges connected to $\eta_1, \cdots, \eta_k \in \mathcal{U}$. Then $4(e, \eta_i)^2 \in \{1, 2, 3\}$ for all i. By (3), no two η's can be connected; so $(\eta_i, \eta_j) = 0$ for $i \neq j$, and $\{\eta_1, \cdots, \eta_k\}$ is an orthonormal set. The projection of e into $\mathrm{span}(\eta_1, \cdots, \eta_k)$ is $e' = \sum_{i=1}^{k} (e, \eta_i)\eta_i$, which is different from e since $\{e, \eta_1, \cdots, \eta_k\} \subseteq \mathcal{U}$ is linearly independent. Therefore,

$$4 = 4(e, e) > 4(e', e') = \sum_{i=1}^{k} 4(e, \eta_i)^2.$$

   Hence $k < 4$.

5. By (4), the only connected graph of an admissible set which contains a triple edge is the Coxeter graph of G_2.

6. If $\{\eta_1, \cdots, \eta_k\} \subseteq \mathcal{U}$ has subgraph ○—○ ··· ○—○, let $\eta := \sum_{i=1}^{k} \eta_i$, then the set

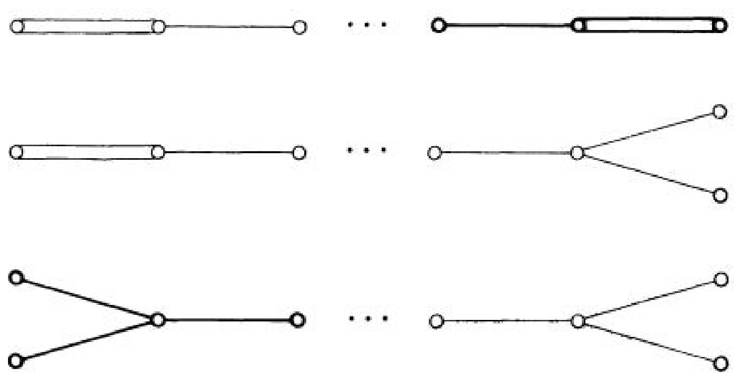
$$\mathcal{U}' = (\mathcal{U} - \{\eta_1, \cdots, \eta_k\}) \cup \{\eta\}$$

   is admissible. Clearly $\mathcal{U}'$ is linearly independent. We have

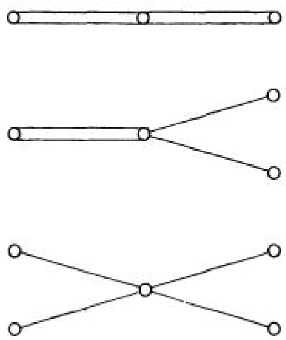
$$(\eta, \eta) = k + \sum_{i=1}^{k-1} (\eta_i, \eta_{i+1}) = k - (k - 1) = 1.$$

   Any $\iota \in \mathcal{U} - \{\eta_1, \cdots, \eta_k\}$ can be connected to at most one of $\eta_1, \cdots, \eta_k$ by (3); so $(\iota, \eta) = 0$ or $(\iota, \eta) = (\iota, \eta_i)$ for some i; we still have $4(\iota, \eta)^2 \in \{0, 1, 2, 3\}$.

7. Γ contains no subgraph of' the form:

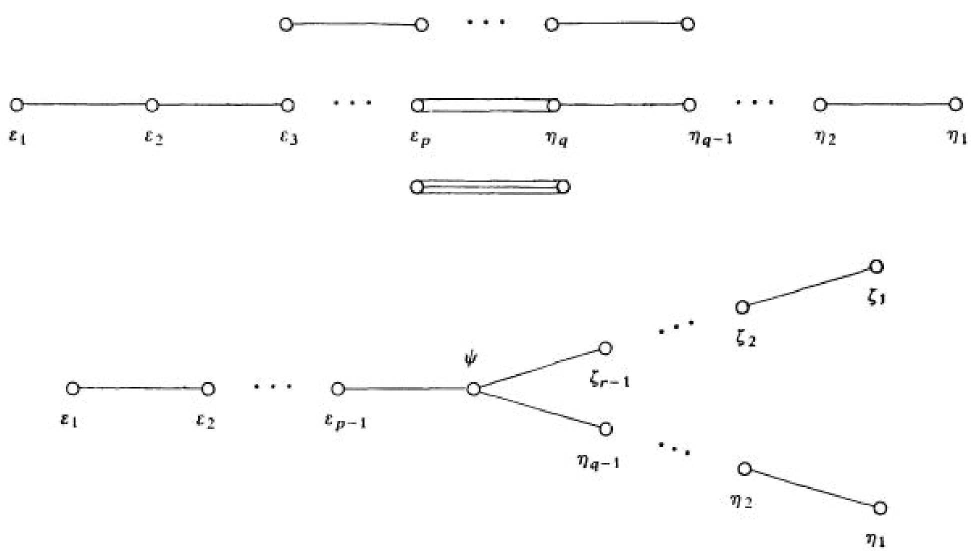


Otherwise, a set of the following graphs is admissible by (6):



which violates (4).

8. Any connected graph Γ of an admissible set has one of the following forms:



9. The only connected Γ of the second type in (8) is the Coxeter graph F_4 or the Coxeter graph $B_n (= C_n)$.

Set $\epsilon = \sum_{i=1}^{p} i\epsilon_i$ and $\eta = \sum_{i=1}^{q} i\eta_i$. Direct calculation shows that

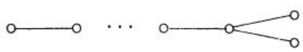
$$(\epsilon,\epsilon) = \sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} i(i+1) = \frac{p(p+1)}{2}, \qquad (\eta,\eta) = \frac{q(q+1)}{2}, \qquad (\epsilon,\eta)^2 = p^2q^2(\epsilon_p,\eta_q)^2 = \frac{p^2q^2}{2}.$$

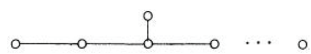
The Schwartz Inequality implies that

$$\frac{p^2q^2}{2} = (\epsilon,\eta)^2 < (\epsilon,\epsilon)(\eta,\eta) = \frac{p(p+1)}{2}\cdot\frac{q(q+1)}{2}.$$

Therefore, $(p-1)(q-1) < 2$. We have $p = q = 2$ (F_4) or $p = 1$ or $q = 1$ (B_n).

10. The only connected Γ of the fourth type in (8) is the Coxeter graph D_n or the Coxeter graph E_n ($n = 6, 7, 8$)





Set $\epsilon = \sum i\epsilon_i$, $\eta = \sum i\eta_i$, $\zeta = \sum i\zeta_i$. Then ϵ, η, ζ are mutually orthogonal, linearly independent, and ψ is not in their span. Let $\theta_1, \theta_2, \theta_3$ be the angles between ψ and ϵ, η, ζ, resp. Then (similar to (4)) the squared length of the projection of ψ onto span(ϵ,η,ζ) is

$$\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 < 1.$$

We have

$$\cos^2\theta_1 = \frac{(\epsilon,\psi)^2}{(\epsilon,\epsilon)(\psi,\psi)} = \frac{(p-1)^2(\epsilon_{p-1},\psi)^2}{(\epsilon,\epsilon)} = \frac{1}{2}(1-\frac{1}{p}).$$

Similarly for θ_2, θ_2. Adding, we get

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

WLOG, assume $1/p \le 1/q \le 1/r$, or $p \ge q \ge r$. If $r = 1$, we get A_n. Otherwise, it is impossible that $r \ge 3$; so $r = 2$. Then $2 \le q < 4$. All possibilities of (p,q,r) are:

$$(p,2,2) = D_n, \qquad (3,3,2) = E_6, \qquad (4,3,2) = E_7, \qquad (5,3,2) = E_8.$$

Finally, from the Coxeter graphs, we easily get the corresponding Dynkin diagrams. □