1.1 Definitions and Examples

1. Verify that F^3 with the cross-product $\times$ is a Lie algebra.
2. Verify that the matrix form of elements of D_ℓ (associate with $s = \begin{bmatrix} 0 & I_\ell \\ I_\ell & 0 \end{bmatrix}$) is $x = \begin{bmatrix} m & n \\ p & q \end{bmatrix}$ where $q = -m^t$, $n^t = -n$, and $p^t = -p$. Then verify the dimension of D_ℓ.
3. (1.1.10)
   (a) Show that A_1, B_1, C_1 are all isomorphic, and D_1 is the one dimensional Lie algebra;
   (b) show that B_2 is isomorphic to C_2;
   (c) show that D_3 is isomorphic to A_3;
   (d) what can you say about D_2?
4. (1.1.11) Verify that the commutator of two derivations of an F-algebra is again a derivation, whereas the ordinary product need not be.

1.2 Ideals and homomorphisms

1. Let L be a Lie algebra. Show that for $z \in L$ and $\delta \in \mathrm{Der}\,(L)$, we have $[\delta, \mathrm{ad}\, z] = \mathrm{ad}\, \delta(z)$. In particular, $\mathrm{ad}\, L$ is an ideal of $\mathrm{Der}\,(L)$.
2. Prove that $\mathfrak{sl}(2, F)$ is a simple Lie algebra if $\mathrm{char}\, F \neq 2$.
3. (optional) Suppose F $= \mathbf{R}$ or $\mathbf{C}$, L is a matrix Lie algebra, and G is the matrix group generated by $\{\exp(tx) \mid \ t \in \mathbf{R}, x \in L\}$. Prove that the linear map ϕ_g, defined by

   $$\phi_g(x) = gxg^{-1} \quad \text{for} \quad x \in L,$$

   is in $\mathrm{Aut}\,(L)$, and $\{\phi_g \mid g \in G\}$ is a normal subgroup of $\mathrm{Aut}\,(L)$.
4. Prove that $\mathfrak{t}(n, F)$ and $\mathfrak{d}(n, F)$ are self-normalizing subalgebras of $\mathfrak{gl}(n, F)$, whereas $\mathfrak{n}(n, F)$ has normalizer $\mathfrak{t}(n, F)$.

1.3 Nilpotent Lie algebras

1. Show that $\mathfrak{t}(n, F)$, the Lie algebra of $n \times n$ upper triangular matrices, is not nilpotent.
2. Prove that L is nilpotent iff $\operatorname{ad} L$ is nilpotent.
3. Prove that the sum of two nilpotent ideal of a Lie algebra L is again a nilpotent ideal. Therefore, L possesses a unique maximal nilpotent ideal.
4. Suppose K is an ideal of a Lie algebra L such that L/K is nilpotent and $\operatorname{ad} x|_K$ is nilpotent for all $x \in L$. Prove that L is nilpotent.

1.4 Solvable Lie Algebras

1. Prove that the Lie algebra $\mathfrak{t}(n, F)$ of $n \times n$ upper triangular matrices is solvable.
2. Let I be an ideal of L. Then each member of the derived series or descending central series of I is also an ideal of L.
3. Prove that L is solvable if and only if there exists a chain of subalgebras

$$L = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_k = 0$$

such that L_{i+1} is an ideal of L_i and such that each quotient L/L_{i+1} is abelian.

4. Let V be a finite dimensional vector space. Show that for any $x, y, z \in \mathfrak{gl}(V)$, we have $\mathrm{Tr}\,([x, y]z) = \mathrm{Tr}\,(x[y, z])$
5. Let $\mathcal{U}$ be a finite dimensional F-algebra, and $\delta \in \mathrm{Der}\,\mathcal{U}$. Prove that for any $n \in \mathbf{Z}^+$ and $x, y \in \mathcal{U}$,

$$(\delta - (a + b) \cdot 1)^n (xy) = \sum_{i=0}^{n} \binom{n}{i} ((\delta - a \cdot 1)^{n-i} x) \cdot ((\delta - b \cdot 1)^i y).$$

6. If $x, y \in \mathrm{End}\,V$ commute, prove that $(x+y)_s = x_s + y_s$, and $(x+y)_n = x_n + y_n$. Show by example that this can fail if x, y fail to commute. [Hint: show first that x, y semisimple (resp. nilpotent) implies $x + y$ semisimple (resp. nilpotent).]
7. Consider the Lie algebra L with a basis $\{x, y, z\}$ defined by:

$$[x, y] = z, \qquad [x, z] = y, \qquad [y, z] = 0.$$

Verify by using Cartan's Criterion on this basis that L is solvable.